Completion Time of Special Kind of GERT-Type Networks with Fuzzy Times for Activities

S.TouSheh Asl#1, S.S.Hasheim#2

#Department of Industrial Engineering, Ardabil Branch, Islamic Azad University, Ardabil, Iran
*Department of Industrial Engineering, Ardabil Branch, Islamic Azad University, Ardabil, Iran

Abstract — Due to uncertainty in realization of some of the project activities, GERT-type networks are used. Also in these networks, duration of activities is non-deterministic. In this research, it is assumed that the probability of activity realization is known. Also, it is supposed that the activity durations can be shown as a positive trapezoidal fuzzy number. Studied networks don’t have any loops. They have one source node and they can have more than one target node. Based on simulation, a new algorithm is developed for acquisition of all sub networks of the project network. It is obvious that the completion time of project is a fuzzy random variable. So, the probability function and cumulative distribution function of completion time of the project are defined. These functions are obtained for each end node of the project. Therefore, mean and variance of completion times can be computed. Finally, an example has been solved using the new algorithm.

Keywords — GERT-type networks, Trapezoidal fuzzy number, Fuzzy random variable, Project completion time.

I. INTRODUCTION

In many of the real-world projects, not only the duration of activities but also their realization time is also non-deterministic. Usually, the duration of activities is estimated using experts’ experience. Expressions like roughly, almost, more or less indicate uncertainty in the estimation of parameters. Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system. In other words, a random variable can have fuzzy values. Studying the completion time of the network is important, because the completion of the project on time can affect the cost. Some new analytical methods for determining the completion time of GERT-type networks have been proposed by [1], [2]. The duration of activities is assumed a random variable in [3], [4]. Gavareshki [5] proposes a new applicable technique for research project scheduling. In this project, nodes are fuzzy and output activities from nodes of network belong to a fuzzy set. This method computes the network completion time as a fuzzy number. Liu et al [6] believe that the traditional GERT networks cannot reflect the characteristics of real-world network problems and uses the triangular fuzzy numbers to formulate the fuzzy GERT model. Fuzzy completion time for alternative stochastic networks has been studied. The intended network contains only nodes with exclusive-or receiver and exclusive-or emitter. Completion time of network is a fuzzy valued random variable. The expected completion time of network is computed as a trapezoidal fuzzy number [7].

If the moment-generating function of random variable of activity time cannot be defined, then the topology equation or maison method cannot be utilized. If we use the topology equation or simplification method, only the moments of network completion time can be calculated. Calculating the high order moments is difficult. So, the first and second moments are usually calculated. But the probability distribution function of network completion time is not computable. If activity duration has special distribution such as Cauchy, then we will not be able to use the topology equation or maison method because the moment-generating function for this distribution is not defined. As a result of this, fuzzy numbers are considered to describe the activity duration. In this paper, it is supposed that the activity duration is shown with a positive trapezoidal fuzzy number.

The paper has the following structure: Section II introduces the definitions, operations and assumptions. Notations are introduced in section III. Section IV introduces the proposed algorithm and describes its steps. Section V describes the stop criteria of algorithm. Section VI explains the ranking method. Section VII illustrates the proposed algorithm using an example. Finally section VIII is devoted to conclusion and recommendations for future studies.
II. DEFINITIONS, OPERATIONS AND ASSUMPTIONS

A. Definitions

In this section, some basic notions of fuzzy theory that have been defined in [8] are introduced.

Definition 1: Let $R$ be the set of real numbers. A fuzzy set $\tilde{A}$ is a set of ordered pairs $\{(x, \mu_A(x)) | x \in X\}$, where $\mu_A(x) : R \rightarrow [0,1]$ and is called membership function of the fuzzy set.

Definition 2: A convex fuzzy set, $\tilde{A}$, is a fuzzy set in which:

For any two TFNs, then:

Definition 3: A fuzzy set $\tilde{A}$ is called positive if its membership function is such that:

Definition 4: Trapezoidal fuzzy number (TFN) is a convex fuzzy set, which is defined as:

For convenience, TFN represented by four real parameters $a, b, c, d$ (where $a \leq b \leq c \leq d$) will be denoted by a trapezoid $(a, b, c, d)$ (Fig. 1).

Fig 1: Trapezoidal fuzzy number (TFN)

Definition 5: A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is called positive TFN if:

0 \leq a \leq b \leq c \leq d.

B. Operations on TFNs

Some of operations can be performed on TFNs. Fuzzy addition; fuzzy multiplication and fuzzy scalar multiplication have been defined in [9].

Fuzzy addition:

Fuzzy multiplication:

Fuzzy scalar multiplication:

Where $\oplus$ = fuzzy addition; $\otimes$ = fuzzy multiplication; $\circ$ = fuzzy scalar multiplication.

C. Assumptions

- The network has a single source node and it can have one or more sink node.
- The network does not contain any loops.
- The probability of network activity realization is known.
- Duration of the network activities is a positive TFN.
- There are several types of nodes in the network.

III. NOTATIONS

The following notations have been used in this paper:

- $D_k$ Duration time of k-th activity (It is a positive TFN),
- $P_k$ Accomplishment probability of k-th activity, given that start node of this activity has been realized,
- $M$ Number of sink nodes,
- $n_i$ Number of paths which start from source node and terminate in i-th sink node,
- $\tilde{T}_{ij}$ The completion time of sub network j-th which terminates in i-th sink node,
- $A_{ij}$ Sub network j-th which terminates in i-th sink node,
- $P_{ij}$ Realization probability of j-th sub network which terminates in i-th sink node,
- $\tilde{T}$ Completion time of the network,
- $P_l$ Realization probability of i-th sink node,
- $P(\tilde{T} = \tilde{T}_{ij}|i)$ Fuzzy random variable probability function of the network completion time,
- $P(\tilde{T} = \tilde{T}_{ij}|i)$ Fuzzy random variable probability function of the network completion time, given that, the node i is sink node,
- $F_{\tilde{T}}(t)$ Fuzzy random variable cumulative distribution function of the network completion time,


\( F_T(t|i) \) Fuzzy random variable cumulative distribution function of the network completion time, given that the node i is sink node.

IV. PROPOSED ALGORITHM

Step 1: \( \bar{T}_S = (0,0,0,0) \)

Step 2: If source is the start node of one activity then suppose that the no. of the activity is k and execute the activity.

\[
\bar{T}_E = \bar{T}_S \oplus \bar{D}_k, \quad (1)
\]

Then the end node of the activity is realized. If this node is one of the end nodes of the network, stop. Otherwise, end node of the activity is start node of other activities. In this case, repeat step 2.

If the output part of start node is PROBABILISTIC, and the node is the start node of several activities then by generating a random number from uniform distribution between 0, 1, the realized activity is identified. If the number of the realized activity is k then execute the activity. So, we have:

\[
\bar{T}_E = \bar{T}_S \oplus \bar{D}_k. \quad (2)
\]

Then the end node of the activity is realized. If this node is one of the end nodes of the network, stop. Otherwise, end node of the activity is start node of other activities. In this case, repeat step 2.

If the output of start node is AND, then all output activities will be realized. The realized activities are divided into Z subsets according to the end nodes. In other words, activities of subset \( L_z \), \( z = 1, ..., Z \) have common end node. So, we will have three cases.

1) If the input of this node is EXCLUSIVE-OR, then \( L_z \) have one member. We will have:

\[
\bar{T}_E = \bar{T}_{S_i} \oplus \bar{D}_k, \quad (3)
\]

The EXCLUSIVE-OR node will be realized. If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

2) If input of end node related to subset \( L_z \) is INCLUSIVE-OR, then we will have:

\[
\bar{T}_E = \bar{T}_{S_i} \oplus \min_{i \in L_z} \{ \bar{D}_i \}. \quad (4)
\]

If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

3) If input of end node related to subset \( L_z \) is AND, then we will have:

\[
\bar{T}_E = \bar{T}_{S_i} \oplus \max_{l \in L_x} \{ \bar{D}_l \}. \quad (5)
\]

If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

Step 3: For newly realized nodes, define the list of all activities which are started from above mentioned nodes. Again, the sum of the realized activities is divided into Z subsets according to the end nodes.

Calculations for each of the subsets depending on input part of end node will be one of the following:

\[
\bar{T}_E = \bar{T}_{S_i} \oplus \bar{D}_i, \quad (6)
\]

\[
\bar{T}_E = \max \{ \bar{T}_{S_i} \oplus \bar{D}_i \}. \quad (7)
\]

\[
\bar{T}_E = \min \{ \bar{T}_{S_i} \oplus \bar{D}_i \}. \quad (8)
\]

\( \bar{T}_{S_i} \) Start time of activity or (activities i)

If one of the end nodes of the network is realized by the realization of the activities of one of the subsets, stop and go to step 4. Otherwise repeat this step. At each step of the algorithm, we record the realized activities. Finally, using the list of the realized activities, the realized sub network can be identified and can be drawn.

Step 4: Obtained \( \bar{T}_E \) for end node i is the completion time of sub network j which terminates in i-th sink node. Occurrence probability of each node is computed and these operations are repeated while the occurrence node is one of the end nodes of the network. Finally, occurrence probability of sub network j which terminates in i-th sink node is obtained.

V. STOP CRITERIA OF ALGORITHM

By repeating the simulation, other sub networks will be obtained. We should check the sum of probabilities of sub networks occurrence in each simulation run. If this summation is equal with one, then stop. Otherwise, repeat the simulation.

The probability function, average, cumulative distribution function and variance will be calculated respectively as follows:

\[
P(\bar{T} = \bar{T}_{ij}) = P_{ij} \quad \forall \ i, j \quad (9)
\]

\[
E(\bar{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left( P_{ij} \bar{T}_{ij} \right) \quad (10)
\]

\[
F_T(t) = \sum_{ij \in I} P(\bar{T} = \bar{T}_{ij}) \quad (11)
\]

\[
\sigma^2 = E(\bar{T} \oplus \bar{T}) \circ [E(\bar{T}) \ominus E(\bar{T})] \quad (12)
\]

Where \( \ominus \) shows the fuzzy subtraction.

We can also calculate \( P(\bar{T} = \bar{T}_{ij}|l) \) and \( F_T(t|i) \).

It is evident that:

\[
E(\bar{T} \oplus \bar{T}) = (a_1, b_1, c_1, d_1). \quad (13)
\]

\[
E(\bar{T} \ominus \bar{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left( P_{ij} \ominus (\bar{T}_{ij} \ominus \bar{T}_{ij}) \right) \quad (14)
\]

A new subtraction operator for positive trapezoidal number was defined such that the subtraction of a positive trapezoidal number is positive [10]. This operator is defined as follows:

\[
d = \max(0, (d_2 - d_1)) \quad ,
\]

\[
c = \max(0, \min(d, (c_2 - c_1))) \quad ,
\]

\[
b = \max(0, \min(c, (b_2 - b_1))) \quad ,
\]

\[
a = \max(0, \min(b, (a_2 - a_1))) \quad ,
\]

\[
\sigma^2 = (a, b, c, d).
\]

Also we can calculate the conditional probability function and conditional cumulative distribution function for each network end node.
VI.ランキング法数による距離法


\[ R(u) = \sqrt{\bar{x}^2 + \bar{y}^2} \]

ここで、\( \bar{x} \) と \( \bar{y} \) は各法数の左端と右端のメンバーシップ関数で、(\( \bar{u}, \bar{u} \))はパラメトリック形式の法数である（定義 1 と 2 を参照）。結果のスカラー値を用いて法数をランク付けします：

1) \( R(u_i) < R(u_j) \) なら \( u_i < u_j \) です。2) \( R(u_i) > R(u_j) \) なら \( u_i > u_j \) です。3) \( R(u_i) = R(u_j) \) なら \( u_i \sim u_j \) です。

定義 1: 一価法数は、メンバーシップ関数 \( u \) が R → I = [0,1] を満たし、以下の条件を満たす：
1) は半連続上向き,
2) \( u(x) = 0 \) は \( x \) が \( [a, b] \) 内に属しない場合に,
3) \( a, b, c, d \) は実数で、\( a \leq b \leq c \leq d \) で
   (a) \( u(x) \) は \( [a, b] \) 内で上向き,
   (b) \( u(x) \) は \( [c, d] \) 内で下向き,
   (c) \( u(x) = 1 \) は \( b \leq x \leq c \) です。

メンバーシップ関数 \( u \) は次のよう表せます:

\[ u(x) = \begin{cases} u_L(x) & a \leq x \leq b, \\ 1 & b \leq x \leq c, \\ u_R(x) & c \leq x \leq d, \\ 0 & \text{otherwise}. \end{cases} \]

定義 2: パラメトリック形式の一価法数 \( u \) は、関数 \( u(r) \) と \( \bar{u}(r) \) を満たし、以下の条件を満たす：
1) \( u(r) \) は \( 0 \leq r \leq 1 \) の範囲で上向き,
2) \( \bar{u}(r) \) は \( 0 \leq r \leq 1 \) の範囲で下向き,
3) \( u(r) \leq \bar{u}(r) \), \( 0 \leq r \leq 1 \) です。

VII.例

Fig. 2 に示されている GERT ネットワークの例です。活動の持続時間は右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されています。TABLE 1 に示すように、活動の持続時間が右端の状態数として表示されて
Using the proposed algorithm in previous sections, we have:

**Step 1:** $\tilde{T}_S = (0,0,0,0)$

**Step 2:** The realization time of node 1 using formula (1) is computed. Then we have:

$\tilde{T}_E = \tilde{T}_S \oplus \tilde{D}_1 = (0,0,0,0) \oplus (3,4,5,6) = (3,4,5,6),$

Because this node is not one of the end nodes of the network, repeat step 2.

**Realized activity:** 1.

**Repeat step 2:** The realization time of node 2 using formula (5) is computed. Then we have:

$\tilde{T}_E = \tilde{T}_S \oplus \max\{\tilde{D}_2, \tilde{D}_3, \tilde{D}_4\} = (3,4,5,6) \oplus (3,4,5,7) = (6,8,10,13)$

The maximum of fuzzy numbers is defined by using the ranking method that was described in VI.

Because this node is not one of the end nodes of the network, go to step 3.

**Realized activity:** 2, 3, 4

**Step 3:** The realization time of node 3 using formula (7) is computed. Then we have:

$L_1 = \{2,3,4\}$

$\tilde{T}_E = \max\{\tilde{T}_{S_2} \oplus \tilde{D}_5, \tilde{T}_{S_3} \oplus \tilde{D}_6, \tilde{T}_{S_4} \oplus \tilde{D}_7\} = (6,8,10,13) \oplus (2,4,7,8) = (8,12,17,21)$

Because this node is not one of the end nodes of the network, repeat step 3.

**Realized activity:** 5, 6, 7

**Repeat step 3:** The realization time of node 4 using formula (6) is computed. Then we have:

$L_1 = \{9\}$

$\tilde{T}_E = \tilde{T}_{S_9} \oplus \tilde{D}_9 = (8,12,17,21) \oplus (3,7,8,9) = (11,19,25,30)$

Because this node is not one of the end nodes of the network, repeat step 3.

**Realized activity:** 9

**Repeat step 3:** The realization time of nodes 5, 6 using formula (7) are computed. Then we have:

$L_1 = \{12\}$

$\tilde{T}_E = \max\{\tilde{T}_{S_{12}} \oplus \tilde{D}_{12}\} = \max\{(11,19,25,30) \oplus (1,5,2,5,4,5,5)\} = (12,5,21,5,29,5,35)$

$\tilde{T}_E = \max\{\tilde{T}_{S_{13}} \oplus \tilde{D}_{13}\} = \max\{(11,19,25,30) \oplus (2,3,5,6)\} = (13,22,30,36)$

Since, nodes 5, 6 are not the end nodes of network, so these nodes are assumed to be start nodes in the next steps.

**Realized activity:** 12, 13

**Repeat step 3:** The realization time of node 8 using formula (8) is computed. Then we have:

$L_1 = \{17\}$

$\tilde{T}_E = \min\{\tilde{T}_{S_{17}} \oplus \tilde{D}_{17}\} = (13,22,30,36) \oplus (3,5,6,7) = (16,27,36,43)$

Because this node is not one of the end nodes of the network, repeat step 3.

**Realized activity:** 14, 17

**Repeat step 3:** The realization time of node 10 using formula (6) is computed. Then we have:

$L_1 = \{19\}$

$\tilde{T}_E = \tilde{T}_{S_{19}} \oplus \tilde{D}_{19} = (16,27,36,43) \oplus (3,5,5,6,8) = (19,5,32,42,51)$

Because, this node is one of the end nodes of the network, stop and go to step 4.

**Realized activity:** 19

Fig 2: GERT network of the example
Step 4:
As a result: $\tilde{T}_2 = \tilde{T}_{10,j}$. The network has two end nodes, so: $M = 2$, $i = 1, 2$ and $j = 1, ..., 13$.
In Fig. 2, assume that node 9 is called end node 1 and node 10 is called end node 2. We have:

$$\tilde{T}_{2,j} = (19.5, 32, 42, 51)$$

Continuing the simulation, by using the list of realized activities sub network can be defined. Fig. 3 shows the above mentioned sub network.

We can also calculate the realized probability of the sub network, so we have:

$$P_{2,j} = 0.4 \times 0.7 \times 0.4 = 0.112$$

In the following, based on the proposed algorithm, a computer program is written in the MATLAB software environment. The output of this program shows the list of the realized activities in tabular form. The realized activities using MATLAB software have been shown in Table II. In Table II, 1 shows the realized activity and 0 shows the unrealized activity. By recognizing the realized activities, required calculations are done. The Table III shows the value of $\tilde{T}_{i,j}$ corresponds to $A_{i,j}$. $P_{ij}$ Shows the occurrence probability of realized sub network that it is shown with $A_{i,j}$.

![Image](image_url)

**Fig 3: One of the identified sub networks**

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<tr>
<th>Activity</th>
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</table>
Using Table III, the realization probability of the end node 1 and end node 2 can be calculated. \( P_1 \) And \( P_2 \) will be as follows:

\[
P_1 = P_{11} + P_{12} + P_{13} = 0.378
\]

\[
P_2 = P_{24} + P_{25} + P_{26} + P_{27} + P_{28} + P_{29} + P_{2.10} + P_{2.11} + P_{2.12} + P_{2.13} = 0.622
\]

The probability function of the network completion time is calculated using formula (9). Table IV shows the network probability function.

**TABLE IV: Probability function of the network**

<table>
<thead>
<tr>
<th>( P(\hat{T} = \hat{T}_{ij}) )</th>
<th>( \hat{T}_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>(10,15,21,26)</td>
</tr>
<tr>
<td>0.06</td>
<td>(17.5,25.5,36.5,46.5)</td>
</tr>
<tr>
<td>0.056</td>
<td>(19,28,38,46.5)</td>
</tr>
<tr>
<td>0.084</td>
<td>(19.5,28,39,47.5)</td>
</tr>
<tr>
<td>0.09</td>
<td>(19.27,5,39,5,50)</td>
</tr>
<tr>
<td>0.12</td>
<td>(18.29,9.5,40,5.51)</td>
</tr>
<tr>
<td>0.084</td>
<td>(20.5,30,41,50)</td>
</tr>
<tr>
<td>0.112</td>
<td>(19.5,32,42,51)</td>
</tr>
<tr>
<td>0.126</td>
<td>(21,30,42,51)</td>
</tr>
<tr>
<td>0.168</td>
<td>(20.32,43,52)</td>
</tr>
</tbody>
</table>

The expected value of network completion time is calculated using Table IV and formula (10).

\[
E(\hat{T}) = 0.1 \odot (10,15,21,26) \\
\odot 0.06 \odot (17.5,25.5,36.5,46.5) \\
\odot 0.056 \odot (19,28,38,46.5) \\
\odot 0.084 \odot (19.5,28,39,47.5) \\
\odot 0.09 \odot (19.27,5,39,5,50) \\
\odot 0.12 \odot (18.29,9.5,40,5.51) \\
\odot 0.112 \odot (19.5,32,42,51) \\
\odot 0.126 \odot (21,30,42,51) \\
\odot 0.168 \odot (20.32,43,52) = (18.534,28.225,38.773,47.678)
\]

Using Table IV and formula (11), we can obtain the cumulative distribution function of the network completion time.

The cumulative distribution function of the network completion time is as follows:

\[
F_{\hat{T}}(t) = \begin{cases} 
0 & t < 10.15,21.26 \\
0.1 & (10,15,21.26) \leq t < (17.5,25.5,36.5,46.5) \\
0.16 & (17.5,25.5,36.5,46.5) \leq t < (19,28,38,46.5) \\
0.216 & (19,28,38,46.5) \leq t < (19.5,28,39,47.5) \\
0.5 & (19.5,28,39,47.5) \leq t < (19.27,5,39,5,50) \\
0.594 & (19.27,5,39,5,50) \leq t < (20.5,30,41,50) \\
0.706 & (20.5,30,41,50) \leq t < (19.5,32,42,51) \\
0.832 & (19.5,32,42,51) \leq t < (21,30,42,51) \\
1 & (21,30,42,51) \leq t
\end{cases}
\]

To compute the variance of the network completion time, first, the values of \( E(\hat{T}) \odot E(\hat{T}) \) and \( E(\hat{T}) \odot E(\hat{T}) \) are calculated using formulas (13) and (14).

\[
E(\hat{T}) \odot E(\hat{T}) = (343.5091,796.6506,1503.3455,2273.1916)
\]
\[ E(\bar{T} \otimes \bar{T}) = (352.557, 819.4875, 1541.5835, 2328.376) \]

Then, the variance of the network completion time is obtained based on formulas (12) and (15).

\[ d = \max(0, (d_2 - d_1)) = \max(0, (2328.376 - 2273.1916)) = 55.1843 \]

\[ c = \max\left(0, \min\left(d, (c_2 - c_1)\right)\right) = \max\left(0, \min\left(55.1843, (1541.5835 - 1503.3455)\right)\right) = 38.2379 \]

\[ b = \max\left(0, \min\left(c, (b_2 - b_1)\right)\right) = \max\left(0, \min\left(38.2379, (819.4875 - 796.6506)\right)\right) = 22.8368 \]

\[ a = \max\left(0, \min\left(b, (a_2 - a_1)\right)\right) = \max\left(0, \min\left(22.8368, (352.557 - 343.5091)\right)\right) = 9.0478 \]

\[ \sigma^2 = (9.0478, 22.8368, 38.2379, 55.1843) \]

For example, the conditional probability function for the end node 1 is obtained using formula (16), the results of which have been shown in Table V.

| \( P(\bar{T} = \bar{T}_{1j}|i = 1) \) | \( \bar{T}_{1j} \) |
|---|---|
| \( \frac{2}{9} \) | (19.5, 28.39, 47.5) |
| \( \frac{1}{3} \) | (21.30, 42.51) |
| \( \frac{4}{9} \) | (20.32, 43.52) |

Using Table V and formula (17), we can obtain the conditional cumulative distribution function of the end node 1.

\[ F_\bar{T}(\alpha|i = 1) = \begin{cases} 
0 & \text{if } (19.5, 28.39, 47.5) \leq \alpha < (21.30, 42.51) \\
(19.5, 28.39, 47.5) & \text{if } (21.30, 42.51) \leq \alpha < (20.32, 43.53) \\
(20.32, 43.53) & \text{if } \alpha < (20.32, 43.53) \\
1 & \text{if } \alpha \geq (20.32, 43.53) 
\end{cases} \]

VIII. CONCLUSION AND RECOMMENDATIONS

In this research, GERT networks have been proposed for analysis of projects in which not only the duration of activities but also their realization time is non-deterministic. An algorithm is proposed to determine the network completion time, so that by repeating the algorithm, other sub networks will be obtained. Then, the mean, variance, probabilistic function and cumulative distribution function of network completion time are calculated as a fuzzy random variable. This study is independent of the fuzzy ranking method. So, we can employ other ranking methods. Obviously, different results can be achieved using different ranking methods. Other researchers can use other ranking methods to compare the results. Also, this problem may be studied when duration of activities are random fuzzy variables. Future researchers can study the problem related to the time cost tread-off with renewable and non-renewable resources for these types of GERT networks.

REFERENCES


