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Finite Element Based Vibration Analysis of an Axially Functionally Graded Nonprismatic Beam

Deepak Ranjan Biswal[#1], AlokRanjan Biswal[#2], RashmiRanjanSenapati*[3]
[#]*Associate Professor,Department of Mechanical Engineering*
DRIEMS Autonomous College, India
[*]*Assistant Professor, Department of Mechanical Engineering*
DRIEMS Autonomous College, India

Abstract -*The present paper deals with vibration analysis of an axially functionally graded non-prismatic beam using the finite element method. A two noded beam element with two degrees of freedom at each node has been considered for the analysis. The varying cross-sectional dimensions with mechanical properties of functionally graded materials have been included in evaluating matrices of structural parts. A polynomial distribution of mass density and modulus of elasticity is assumed in the present study. The convergence study has been carried out with the existing available results.*

Keywords - *Finite element analysis, Axially FG beam, Nonprismatic beam.*

I. INTRODUCTION

Functionally graded (FG) materials are among the most advanced materials that play a significant role in various applications such as automotive industries, aerospace, and machine components. The physical properties change continuously in these materials' preferred direction, which provides continuous stress and strain in the grading directions. By employing FG material, in comparison with laminated composites, the structural systems lead to eliminating stress concentration and improving the strength and toughness of the material. Several literature studies dealt with FG beams whose mechanical properties vary in thickness directions [1-4]. There is relatively less work on axially FG beams commenced whose properties vary along the beam axis. Due to the simultaneous variation of mass density, modulus of elasticity, and the beam axis area, the governing differential equation, and its solution could hardly be found or even impossible to obtain. Hence numerical technique is essential. A semi-inverse method for the governing differential equation solution has been considered [5, 6]. Further, a semi-inverse method of the uniform beam with constant mass density is used to study the free transverse vibration and stability analysis of axially FG supported beams [18].

The free vibration analysis of an FG ordinary twisted Timoshenko beam of cantilever type was investigated. The simultaneous effects of power-law index and pre twist angle on first natural frequency were conducted and observed that it was marginal [8].The position of the natural surface of the FG beam was obtained, and its influence on the deflection of the beam under UDL was studied [9].The numerical calculations for the natural frequencies of FG supported beams were presented. The first order Timoshenko beam theory and third-order shear deformation theory were applied for the analysis of FG beam [10]. The effects of material properties on the nonlinear dynamic behavior of the FG beam were discussed. The system's frequency response equation was presented, and the effects of different parameters on the response of the system were investigated [11].

On the other hand, several investigations have been studied on beams' free vibration with non-uniform cross-section. The governing equation for this problem is still with varying flexural rigidity and distributed mass. Rayleigh's optimization technique has been used to obtain an approximate solution of natural frequencies and buckling loads for a non-uniform beam [12] subjected to varying loading conditions. The non-uniform tapered rod has been investigated using the Rayleigh-Ritz and Lagrange multiplier method [13]. The exact analytical solutions of the longitudinal and transverse vibration of rods and beams have been found by converting the governing equation of motion to a differential equation. These differential equations have been solved by Bessel functions [19]. A numerical method has been encompassed for finding the natural frequencies of non-uniform beams, which is based on the fact that these nonhomogeneous beams can be partitioned into multi homogeneous beams [15]. The natural frequencies and mode shapes of nonhomogeneous rods and beams can be found using the functional perturbation method considering the random probability distribution of material properties [16].

Although several investigations have been carried out for FG and nonprismatic beams, there is still a gap in the direction of varying material properties. Further varying material properties in the axial direction with arbitrarily profile cross-section has not been discussed yet. The present article focuses mainly on varying cross-sectional area with a simultaneous variation of material properties in the axial direction. The static and dynamic analysis for the proposed beam has been carried out using the finite element method.

II. MATHEMATICAL FORMULATION

The beam's mathematical formulation includes modeling the cross-section of the beam and the axially functionally graded beam's modeling. The cross-section profile of the proposed beam is represented as

$$A_b(x) = A_0\left(1 - c_b\,\frac{x}{L}\right)\left(1 - c_h\,\frac{x}{L}\right) \qquad (1)$$

The geometric profile of the modeled beam is shown in Figure 1. The length of the beam is L_b. The beam's width and height taper ratios are denoted as cb and ch, which could vary in the range of $0 \leq c_b \leq 1$ and $0 \leq c_h \leq 1$. When $c_b = c_h = 0$, the beam will become a uniform one, and when $c_b = c_h = 1$, the beam would taperto a point at $x=L$, which is a theoretical limit.

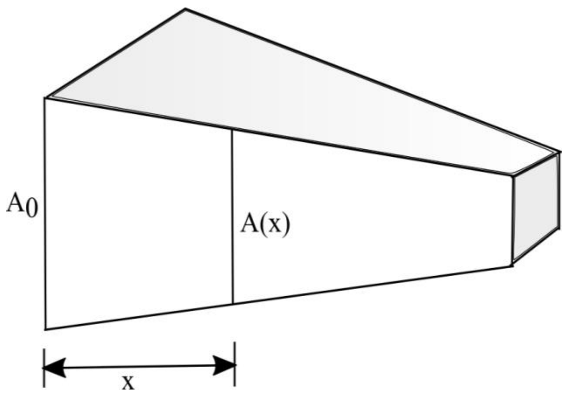

Figure 1. Geometric profiles of beam

The beam is modeled as FG, i.e., non-homogeneity in material properties (such as density, Young's modulus) in the axial direction. The following mathematical expression has been proposed to determine such FG properties of the beam in the axial direction

$$E(x) = \sum_{i=0}^{m} a_i X^i \quad and \quad \rho(x) = \sum_{i=0}^{r} b_i X^i \qquad (2)$$

Where *a*and b are coefficients of density and young's modulus, respectively.

A. Displacement field

The prismatic beam element with degrees of freedom at each node is shown in figure 2. Each nodal point of the element experiences two degrees of freedom, i.e.,*v* and *θ*, which are supposed to act at each node of the beam element.

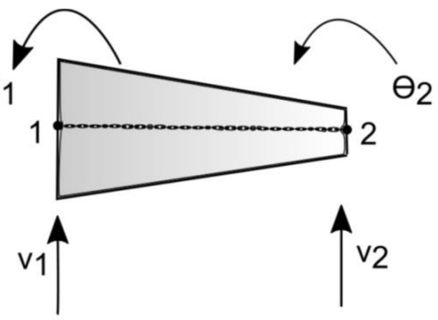

Figure 2. Nodal degrees of freedom of beam element

The displacement field of the beam in x, y, and z-direction can be written as

$$\begin{aligned} u(x,y,z,t) &= -z\theta(x,t) = -z\left(\frac{\partial w}{\partial x}\right) \\ v(x,y,z,t) &= 0 \\ w(x,y,z,t) &= w(x,t) \end{aligned} \qquad (3)$$

Where *v*, *w*is the time-dependent lateral and transverse displacements along *y*, *z*-axes, respectively. The terms *w(x, t*) is the transverse displacement of any point in the midplane (z=0). The term θ is the rotation of the midplane about the *y*-axis, whereas *t* denotes the time. The axial displacement at any point in the midplane (z=0) is neglected as its effect is negligible compared to transverse displacement. Moreover, as output power is greatly influenced by bending strain; hence membrane strain is neglected for the above expressions.

B. Shape function

The displacement field could be interpolated in terms of degrees of freedom of nodes and shape functions based on the concept of FEM as

$$\{w\} = [N_w]\{q_w\} \qquad (4)$$

Here q_w and N_w signify the nodal degrees of freedom and the bending shape functions, respectively. The accuracy of the result is governed by how well the shape function is selected. The shape function is interpolated in matrix form as

$$N_w = \left[1 - \frac{3x^2}{l} + \frac{2x^3}{l^3} \quad x - \frac{2x^2}{l} + \frac{x^3}{l^2} \quad \frac{3x^2}{l^2} - \frac{2x^3}{l^3} \quad -\frac{x^2}{l}(5)\frac{x^3}{l^2}\right]$$

Where *l* is the length of the beam element.

C. Governing equation

Using Hamilton's principle, the dynamic equations of motion for energy harvesting system is represented as

$$\partial\Pi = \int_{t_1}^{t_2} [\partial(E_K - E_P + W)]dt = 0 \qquad (6)$$

EK is kinetic energy. EP is the total electromechanical enthalpy, and W is the total work done by the external mechanical and electrical force. The terms t_1 and t_2 represent the initial and final time. Theexpressions for the above terms can be written as

$$E_k = \frac{1}{2}\int_{V_b} \rho_b \dot{w}^T \dot{w} dV_b \tag{7}$$

$$E_p = \frac{1}{2}\int_{V_b} \varepsilon_1{}^T \sigma_1 dV_b \tag{8}$$

$$W = \sum_{i=1}^{nf} r(x_i)Q(x_i) \tag{9}$$

V is the volume, ρ is the density, r is the displacement vector, Q is the magnitude of force, and suffix b represents the beam material.The equation (6) can be rewritten as

$$\partial \Pi = \begin{bmatrix} \int_{t_1}^{t_2}\int_{V_b} \frac{1}{2}\rho_b \partial \dot{w}^T \dot{w} dV_b - \int \frac{1}{2}\partial\varepsilon_1{}^T \bar{c}_{11}^E \varepsilon_1 dV_b \\ + \sum_{i=1}^{nf} \partial r(x_i)Q(x_i) \end{bmatrix} = 0 \tag{10}$$

From equation (10), the elemental matrices of the beam can be expressed as

$$[M_b^e] = \int_0^{L_b} [N_w]^T \rho_b A_b [N_w] dx \tag{11}$$

$$[K_b^e] = \int_0^{L_b} \left[\frac{\partial[N_\theta]}{\partial x}\right]^T E_b I_b \left[\frac{\partial[N_\theta]}{\partial x}\right] dx \tag{12}$$

The equation of motion of the piezolaminated discretized structure is represented by

$$\left[M_b^e\right]\{\ddot{r}\} + \left[K_b^e\right]\{r\} = \{Q^e\} \tag{13}$$

The stiffness matrices, mass matrices, and coupling matrices have been evaluated by numerical integration using two points Gauss quadrature. The system should have some supplementary structural damping, which needs to be taken into account. By using the proportional damping method, the damping ratio is predicted from the computed fundamental frequency as

$$C_b^e = \alpha\left(M_b^e\right) + \beta\left(K_b^e\right) \tag{14}$$

Where α and β are found out from

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}, \quad i=1,\ 2,\ 3 \ldots n \tag{15}$$

Where ζ_i is the damping ratio of the structure. Hence incorporating equation (15) in equation (13), the final equation of motion for piezolaminated beam is found as

$$\left[M_b^e\right]\{\ddot{r}\} + \left[C_b^e\right]\{\dot{r}\} + \left[K_b^e\right]\{r\} = \{Q^e\} \tag{16}$$

After assembling the elemental matrices, the global set of equation become

$$[M_B]\{\ddot{r}\} + [C_B]\{\dot{r}\} + [K_B]\{r\} = \{Q\} \tag{17}$$

The terms MB is the global mass matrices of the beam, KB is the global stiffness matrices for beam and CB is the global proportional damping matrix.

D. *State-space representation*

This method is used to develop the system's uncoupled governing equations of motionin terms of principal coordinates. It canbe achieved by introducing a linear transformation between the generalized coordinates *r(t)* and the principal coordinates *g(t).* By using the transformation matrix, the displacement vector r(t) can be approximated as

$$r(t) = x(n)g(t) \tag{18}$$

Where *x(n)* is represented as the modal matrix containing the eigenvectors representing the vibratory modes. By considering the modal damping, the decoupled equation can be written as

$$\left\{\ddot{g}_i(t)\right\} + 2\zeta_i\omega_i\left\{\dot{g}_i(t)\right\} + \omega_i{}^2\{g_i(t)\} = x(n)^T\{Q\} \tag{19}$$

Using the state-space form, equation (19) can be expressed as

$$\{\dot{X}\} = [A]\{X\} + B\{u\} \tag{20}$$

Where

$$[A] = \begin{bmatrix} [0] & [I] \\ [-\omega_i^2] & [-2\zeta_i\omega_i] \end{bmatrix}, \quad [B] = \begin{bmatrix} [0] \\ [-x(n)^T Q] \end{bmatrix},$$
$$\{\dot{X}\} = \begin{Bmatrix} \dot{g}(t) \\ \ddot{g}(t) \end{Bmatrix} \quad and \quad \{X\} = \begin{Bmatrix} g(t) \\ \dot{g}(t) \end{Bmatrix} \tag{21}$$

The terms *A*, *B*, *X*, and *u* are the system matrix, input matrix, state vector, and the input vector. The output equation for the sensor can be written as

$$\{y\} = [C]\{X\} \tag{22}$$

Where *C*isis represented as the output matrix.

III. RESULT AND DISCUSSION

A.Structural validation of prismatic beam

The present code is validated by considering a cantilever beam of the rectangular cross-section.

The dimensions of the beam are (800×60×6) mm. Material properties considered are E = 210 GPA, ρ = 7850 kg/m^3 and μ = 0.3. The applied load at the free end is considered as 1N. The beam is divided into ten number of FEs. The exact solution for natural frequencies for Euler-Bernoulli beam has been found out as

$$\omega = (\beta L_b)^2 \sqrt{\frac{EI}{\rho A L_b^4}} \tag{23}$$

Where EI=Flexural rigidity, ρA=Mass per unit length, L_b=length of the beam, and the values of 'βL_b' for the first four natural frequencies are 1.875, 4.694, 7.854, and 10.995. The natural frequencies are calculated using the present code developed and compared with the exact solution and represented in Table 1. Table 1 shows that the results obtained by using the developed MATLAB code are in good agreement with the theoretical values using equation (23).

Table.1 Validation of natural frequencies of a prismatic beam

Natural frequency(Rad/sec)	Present code	Exact
ω_1	43.055	43.013
ω_2	269.904	269.305
ω_3	756.036	755.976
ω_4	1482.347	1481.231

B. Structural validation of non-prismatic beam

In this section, a non-prismatic (taper ratio (α) =0.2, 0.4) homogeneous cantilever beam of the rectangular cross-section is considered for structural validation of present formulations. The first four nondimensional fundamental frequencies are calculated using the presently developed computer code and compared with the existing results, as listed in Table 2. The present results are corroborated with the existing results obtained [17]. To facilitate the presentation of results, the nondimensional fundamental frequency can be introduced as

$$\overline{\omega} = \omega \sqrt{\frac{\rho_0 A_0 L_b^4}{E_0 I_0}} \tag{24}$$

The subscript 0 designates the value of parameters at the fixed end of the beam.

Table.2 Validation of natural frequencies of the non-prismatic beam

Natural Frequency (rad/sec)	shabha and Rajasekaran[17]	Present code
α=0.2		
ω_1	18.21	19.02
ω_2	50.48	52.84
α=0.4		
ω_1	15.82	16.23
ω_2	44.02	45.47

C. Frequency of axially FG nonprismatic beams

The first three fundamental frequencies for different taper values, such as (0.2, 0.4, 0.6, and 0.8), are presented in Table 3-5. The table shows that with increase in height taper (ch), the natural frequency decreases by keeping width taper (c_b) remains constant. Further, with an increase in the natural frequency decreases by keeping c_h remains constant.

Table.3. 1st natural frequency of axially FG nonprismatic beam

1st natural frequency (Rad/sec)				
	c_h			
c_b	0.2	0.4	0.6	0.8
0.2	40.07	35.08	28.43	19.95
0.4	40.34	34.87	27.84	19.22
0.6	40.30	34.27	26.85	18.18
0.8	39.47	32.83	25.11	16.59

Table.4. 2nd natural frequency of axially FG nonprismatic beam

2nd natural frequency (Rad/sec)				
	c_h			
c_b	0.2	0.4	0.6	0.8
0.2	251.14	219.90	178.29	125.17
0.4	252.83	218.60	174.61	120.63
0.6	252.59	214.86	168.45	114.12
0.8	247.44	205.88	157.57	104.17

Table.5. 3rd natural frequency of axially FG nonprismatic beam

3rd natural frequency (Rad/sec)				
	c_h			
c_b	0.2	0.4	0.6	0.8
0.2	703.23	615.75	499.40	350.85
0.4	707.95	612.17	489.19	338.23
0.6	707.31	601.79	472.07	320.09
0.8	693.03	576.89	441.81	292.35

The first five-mode shapes of an axially FG nonprismaticmodeled beam with c_b=0.4 and c_h=0.6 has been presented in Fig 3.From the figure, it has been observed that the absolute amplitude for 1st mode has more tip displacement than others. Similarly, the frequency response has been presented in Fig 4. For this, the driving frequency varies from0-600 Rad/ Sec. It has been observed that the peak response for the modeled beam has a higher value in 1st resonant frequency than the 2nd resonant frequency.

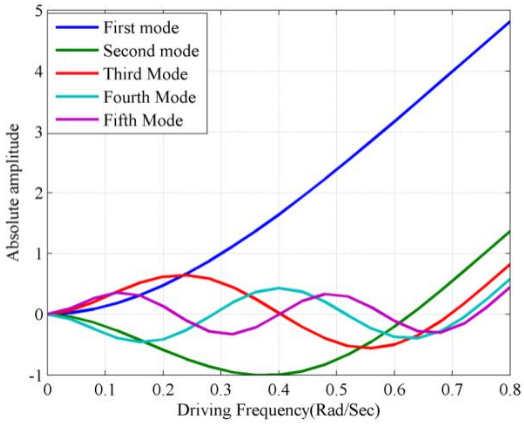

Fig 3 First five mode shapes of axially FG beam with c_b=0.4 and c_h=0.6.

D. Frequency response

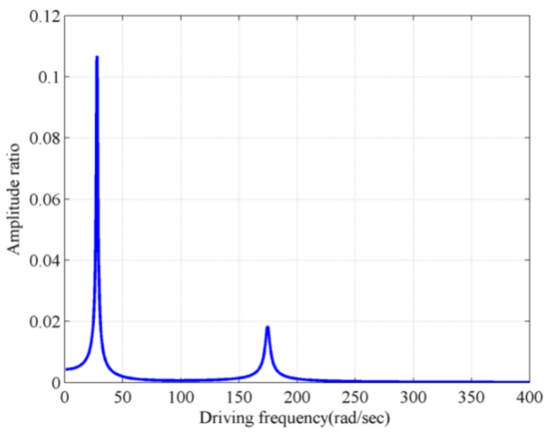

Fig 4. The frequency response of axially FG Beam with c_b=0.4 and c_h=0.6.

The time-domain response of axially FG beam for arbitrarily varying taper values such as c_b=0.4 & 0.6 and c_h=0.4 & 0.6 has been considered. The time interval for the response has been taken as 2 Sec. From the figure, it has been observed that the response goes on dies out after some time. This is because the amplitude decreases with an increase in time due to structural damping in the modeled beam.

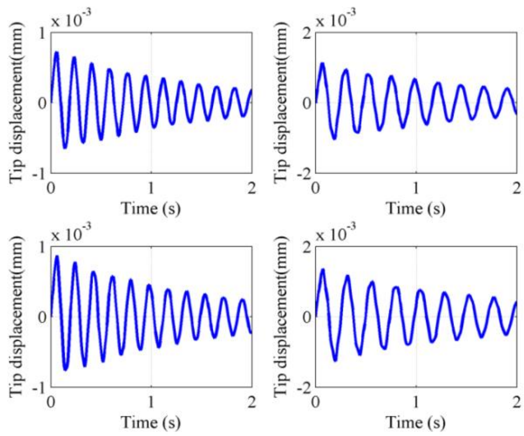

Fig 5. The time-domain response of an axially FG beam with c_b=0.4 and c_h=0.6.

IV. CONCLUSION

The present article focused on vibration analysis of an axially FG non-prismatic, i.e., tapered beam with finite element (FE) based modeling. The beam is modeled using the Euler-Bernoulli beam theory. Two nodded beam elements with two degrees of freedom at each node are considered to solve the governing equation. From the analysis, it is observed that the non-prismatic beam has less frequency than the prismatic beam due to the uniform distribution of strain from the analysis. Further frequency response and time domain response for different taper values has been discussed.

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