SSRG International Journal of Mechanical Engineering Volume 5 Issue 1, 26-30, January 2018
ISSN: 2348 – 8360 /doi:10.14445/23488360/IJME-V5IP105

Steady-State Motions of Machines with Finite Degree of Freedom Influenced by Position and Velocity Depending Forces

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Abstract — *An algorithm for the solution of non-linear differential equations describing the steady-state motion (T-periodical) of the mechanical systems with one of the finite numbers of degrees of freedom is presented. The initial approximation of the solution is obtained in a poly-harmonic form, one of the most significant merits of the proposed approach. Criteria for estimating the number of harmonics in the stationary solution minimizes the approximation's iterations.*
A first-order linear differential equation solves the problem defined in the paper with one degree of freedom. For the systems with a finite number of degrees of freedom, the method's application leads to an extreme task with many variables.

Keywords — *the steady-state motion of machine's units; solution in harmonics; non-linear equations.*

I. INTRODUCTION

There are many successive approaches and methods for the solution of one and multidimensional non-linear differential equations. Widely known are the methods of a small parameter, harmonic linearization, Bubnov-Galerkin's, Lyapunov – Linsted's, asymptotically of Bogoliybov-Mitropolsky, and others [1], [2], [3]. Many of them differ in accuracy, convergence, difficulty, range of applicability, and so on. Practically there is no universal algorithm or method for a solution in a poly-harmonically form applicable to any nonlinearities, respectively equations.

This work proposes an approach and methodology for solving the classic equations of motion of mechanical /machine's unit/ systems with one of the finite numbers of the degree of freedom influenced by generalized forces depending on the positions and velocities. In this approach, it is very interesting that the non-linear equation(-s) solution in its initial approximation is obtained in poly harmonically form without no further solution of equations of a higher order.

II. NOMENCLATURE

$q(q_i)$ – generalized coordinate $q=1$ or,

$q_i=(1....n)$;

Q_M, Q_r – driving, restoring forces;

$Q(Q_i)=Q_M-Q_r$ – generalized force;

$q=q(t+T)$ – periodical motion;

λ_{ij} =IIEII – the symbol of Kronecer, or unit matrix;

$m(q)$ – generalized mass parameter(-s).

III. THEORETICAL CONSIDERATION

The differential equation describing the motion of a mechanical system with one degree of freedom is well known and has the form [4], [5]:

$$m(q)\ddot{q}(t)+\frac{1}{2}\dot{q}^2(t)\frac{dm(q)}{dq}= \\ =Q_M(q,\dot{q})-Q_r(q,\dot{q})\equiv Q_r \tag{1}$$

where

$$m(q)=m(q+q_{TM}).$$

The reduction of mass and inertia characteristics of a wide range of machines is a function of the position. Its principal character is depicted in Fig.1, where the ratio between the maximum of the variation $\Delta m_{\max}$ and the constant part m_0 is within the limits

$$0\le\frac{\Delta m_{\max}}{m_0}\le 0,5,$$

or $$m(q)=m_0+\Delta m(q+q_{TM}).$$

Within the steady-state (stationary) motion, a T–periodic solution is sought [4], [5] in the form

$$q(t)=q(t+T)\equiv\dot{q}_0 t+\gamma(t) \\ \dot{q}(t)=\dot{q}_0+\dot{\gamma}(t);\ \ \ddot{q}=\ddot{\gamma}(t);\ \ (\cdot)\equiv\frac{d}{dt}. \tag{2}$$

Assuming that the reduced driving $Q_M(q,\dot{q})$ and $Q_r(q,\dot{q})$ resisting forces are represented by the average integral values of the small variations that depend on the position and velocity [2], [4], or

$$Q_M(q,\dot{q})=Q_{M_0}(\dot{q})+\Delta Q_M(q,\dot{q}); \\ Q_r(q,\dot{q})=Q_{r_0}(\dot{q})+\Delta Q_r(q,\dot{q}). \tag{3}$$

Decomposition of (3) in a series of Mac-Loren in the vicinity of $\dot{q}_0$ and $(q_0 t)$ represent only the linear members – i.e.:

$$(4)\quad \left| \begin{array}{l} Q_M(q,\dot{q}) = Q_{M0}(\dot{q}_0) + v\dot{\gamma}(t) + \Delta Q_M(q_0 t,\dot{q}_0) + \\ + \dfrac{d\Delta Q_M(q,\dot{q})\gamma(t)}{dq} + \dfrac{d\Delta Q_M(q,\dot{q})\dot{\gamma}(t)}{d\dot{q}} \\ Q_r(q,\dot{q}) = Q_{r0}(\dot{q}_0) + w\dot{\gamma}(t) + \Delta Q_r(q_0 t,\dot{q}_0) + \\ + \dfrac{d\Delta Q_r(q,\dot{q})\gamma(t)}{dq} + \dfrac{d\Delta(q,\dot{q})\dot{\gamma}(t)}{d\dot{q}} \end{array} \right. ,$$

where

$$v = \tan\theta \equiv \frac{dQ_{M0}(\dot{q})}{d\dot{q}};\; w = \tan\eta = \frac{dQ_{r0}(\dot{q})}{d\dot{q}}.$$

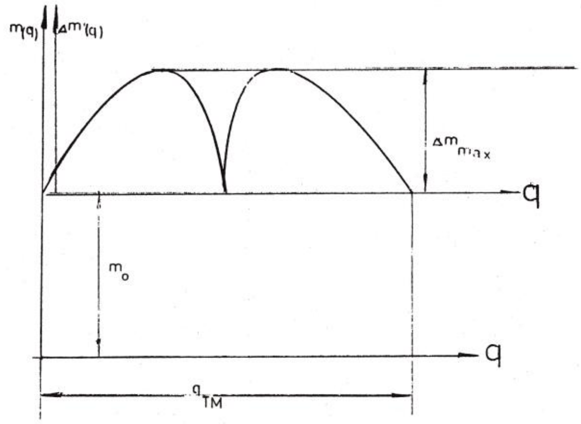

Fig 1: The reduced mass characteristic, as a function of q

Fig.2 depicts the functions $Q_{M0}(\dot{q})$ and $Q_{r0}(\dot{q})$ the average angular velocity , corresponding to the equivalence of the kinetic energy of the driving and resisting forces within the period that determines the machine unit's steady-state motion.

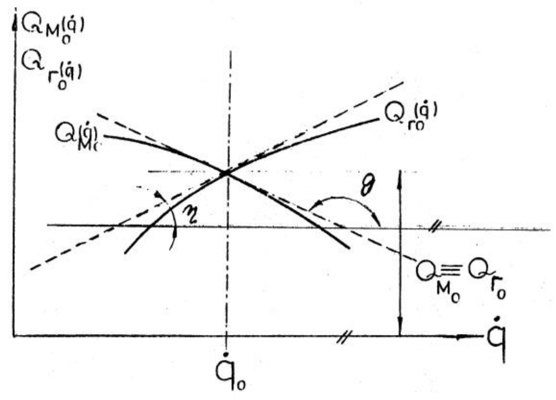

Fig.2: The average integral values $Q_{M0}(\dot{q})$ **and** $Q_{r0}(\dot{q})$

After replacing (2) and (4) in (1) and neglecting the small members of a higher order, one obtains [4]

(5)

$$\left[m_0 + \Delta m(q)\right]\left[\frac{d\dot{q}_0}{dt} + \ddot{\gamma}(t)\right] + \frac{1}{2}\dot{q}_0^2\frac{dm(\dot{q}_0 t)}{d(q_0 t)} = \\ = (v-w)\dot{\gamma}(t) + Q_{M0}(\dot{q}_0) - Q_{r0}(\dot{q}_0) + \\ + \Delta Q_M(q_0 t,\dot{q}_0) + \Delta Q_r(q_0 t,\dot{q}_0),$$

or

$$(6)\quad \left| \begin{array}{l} m_0\dfrac{d\dot{q}_0}{dt} = Q_{M_0}(\dot{q}) - Q_{r_0}(\dot{q}) \equiv 0 \therefore \dot{q}_0 = \\ = \dot{q}_{0st} \equiv const \\ \ddot{\gamma}(t) - \dfrac{v-w}{m_0 + \Delta m(q_0 t)}\dot{\gamma}(t) = \\ = \left[\Delta Q_M(q_0 t,\dot{q}_0) - \Delta Q_r(q_0 t,\dot{q}_0) - \right. \\ \left. -0{,}5\dot{q}_0^2 . dm(q_0 t)/d(q_0 t)\right]\dfrac{1}{m_0 - \Delta m(q_0 t)} = \\ = f_2(t) \end{array} \right.$$

The condition of sustainable cooperation of the motor (the power) and the working machine is well known [4], [5] and could be described as follows:

$$\frac{d}{d\dot{q}}\left[Q_{r0}(\dot{q}) - Q_{M0}(\dot{q})\right] \square\ 0,$$

from which it follows that

$$m_0 + \Delta m(q_0 t) > 0, \quad \frac{v-w}{m_0 + \Delta m(q_0 t)} < 0.$$

The second equation in (6) is represented by

$$(7)\quad \begin{array}{l} \dot{x}(t) + f_1(t)x(t) = f_2(t), \\ f_1(t) = -\dfrac{v-w}{m_0 + \Delta m(q_0 t)} \end{array},$$

where $\dot{x} = \ddot{\gamma}(t)$ $x = \dot{\gamma}(t)$; $f_1(t+T)$ and $f_2(t+T)$ are the functions in front of $\dot{\gamma}(t)$ and right of the (6).

The equation (7) is a linear non - homogeneous equation of the first order whose solution has the form [5]

$$(8)\quad \dot{q}(t) = \dot{q}_0 + e^{-\int f_1(t)dt}\left[\int e^{\int f_1(t)dt}.f_2 dt + C\right]$$

The functions $f_1(t)$, $f_2(t)$ are periodic, satisfying the Dirihle criteria, and can be expanded in Fourier series to choose the number of harmonics.

When the variation of the reduced mass characteristic $\Delta m(q)$ is relatively small

$\Delta m_{\max} \approx 0,2m_0$ (Fig. 1), then the second equation of (6) is represented by

$$\dot{x}(t) + k_1 x = \frac{1}{m_0}\Big[\Delta Q_M(q_0 t, \dot{q}_0) - -\Delta Q_r(q_0 t, \dot{q}) - 0,5\frac{dm(q_0 t)}{d(q_0 t)}\Big] = f_3(t) \quad (9)$$

where

$$k_1 = \frac{w-v}{m_0};\ f_3(t) = f_3(t+T) = = \sum_{s=o}^{\infty} L_s \sin(spt + \delta_s);\ p = \frac{2\pi}{T}$$

In this case, the solution of the differential equation (9) can be represented in the following form

$$\dot{q}(t) = \dot{q}_0 + \frac{\sum_{s=o}^{n} l_s \sin\left(spt + \delta_s + \delta_s^*\right)}{k_1^2 + k_1 p \sum \lambda_{is} s^2}, \quad (10)$$

where

$$l_s = L_s\sqrt{k_1^2 + s^2 p^2}:\ \ \delta_s^* = arctg\frac{ps}{k_1}:\ \lambda_{is} = \|E\| .$$

If the generalized force in the partial case is a function of the position and velocity of a detachable type like this:

$$Q(q_0, \dot{q}) = f_1(q) + \dot{q}^2 f_2(q), \quad (11)$$

so the equation (1) is reduced to a first-order linear equation concerning squared generalized machine's velocity, namely

$$\frac{d\dot{q}^2}{dq} + \dot{q}^2\left[\frac{1}{m(q)}\frac{dm(q)}{dq} - 2\frac{f_2(q)}{m(q)}\right] = 2\frac{f_1(q)}{m(q)}, \quad (12)$$

which after integration has the form [5]

$$\dot{q}^2 = \frac{1}{m(q)}\left\{e^{2\int\frac{f_2(q)}{m(q)}dq}\left[2\int\frac{f_1(q)dq}{e^{2\int\frac{f_2(q)}{m(q)}dq}} + C\right]\right\}. \quad (13)$$

Let us assume that,

$$\sigma(q) = e^{-2\int\frac{f_2(q)}{m(q)}dq}, \quad (14)$$

(13) is transformed into

$$\dot{q}^2 = \frac{1}{m(q)}\left\{e^{2\int\frac{f_2(q)}{m(q)}dq}\left[2\int\sigma(q)f_1(q)dq + C\right]\right\} \quad (15)$$

Under initial condition $t=0$; $q=0$; $\dot{q}=\dot{q}_0$ and $m(0)=m_0$, then [5]

$$\begin{gathered}\frac{C}{2} = \sigma(0)\frac{m_0}{2}\dot{q}_0^2 - \left|2\int\sigma(q)f_2(q)dq\right|_{q=0} \\ \sigma(q)m(q)\frac{\dot{q}^2}{2} - \sigma(0)m_0\frac{\dot{q}_0^2}{2} = \int_0^q \sigma(q)f_1(q)dq\end{gathered} \quad (16)$$

It is noteworthy that the equation (16) corresponds to the first integral of (1) for a conservative system with one degree of freedom when the force is a function of the position $Q=Q(q)$ then $\sigma(q)=1$, respectively $f_2(\dot{q})=0$.

When the generalized force is a function only of the velocity

$$Q(\dot{q}) = Q_M(\dot{q}) - Q_r(\dot{q}),$$

then in equation (6), the expression

$$Q_{M0}(\dot{q}_0) \equiv Q_M(\dot{q}_0):\ Q_{r0}(\dot{q}_0) \equiv Q_r(\dot{q}_0): :\Delta Q_M(q_0 t, \dot{q}_0) = \Delta Q_r(q_0 t, \dot{q}_0) = 0$$

and the algorithm for obtaining the law of motion in the steady-state model remains the same.

The steady-state motion of a wide range of machine aggregates (unites) with a finite number of degree of freedom is described by a system of differential equations of the first order of the type:

$$\frac{dx}{dt} = f(x,t), \quad (17)$$

where

$x = x\{x_1, x_2, ...x_n\}$ – vector of unknown values;

$f = \{f_1, f_2, ..., f_n\}$ – vector of no-linear functions;

$f(x,t) = f(x, t+T)$.

The search for a steady-state motion of the machine is sought in the area for arbitrary initial conditions . There is only one continuous solution $x = x(x_0, t)$, whose derivatives are persistent with greater $k - \left[\left.\frac{d^k x}{dt^k}\right|_{t=0}\right]$.

A T-periodical solution for (17) is sought in the form [6]

$$\begin{gathered} f(x,t) = f(x,t+T) \quad \forall t \\ \xi(\xi_o, t) = \xi(\xi_o, t+T) \quad \forall t \end{gathered} \quad (18)$$

In accordance with the prerequisites made, it is assumed that,

$$\xi(\xi_0, t) = \xi_0 \sum_{i=1}^{n}\left(\xi_{ci}\cos\frac{2\pi}{T}t + \xi_{si}\sin\frac{2\pi}{T}t\right), \quad (19)$$

where **n** is a preselected number of harmonics to obtain the required accuracy, and the amplitudes of the harmonics in the Fourier series satisfy the following algebraic system [6], [8], [9]:

$$(20)\quad \left| \begin{array}{ll} (-1)^{\frac{k}{2}} \sum_{i=0}^{n} \left(i\frac{2\pi}{T} \right)^{k} \xi_{ci} = \xi_0^{(k)} & k = (0,2,4....2n) \\ (-1)^{\frac{k}{2}+1} \sum_{i=0}^{n} \left(i\frac{2\pi}{T} \right)^{k-1} \xi_{si} = \xi_0^{(k-1)} & k = (2,4....2n) \end{array} \right.$$

From the system of differential equations (17), one obtains

$$(21)\quad \left| \begin{array}{l} \frac{dx}{dt}_{|(t=0)} = f(x,0) = \frac{dx}{dt}(x_0) \\ \frac{d^2x}{dt^2}_{|(t=0)} = \frac{\partial f(x,0)}{\partial x}\frac{\partial x}{\partial t}_{|(t=0)} + \frac{\partial f(x,0)}{\partial t} = \frac{d^2x}{dt^2}(x_0) \\ \frac{d^{2n}x}{dt^2}_{|(t=0)} = ... = \frac{d^{2n}x}{dt^2}(x_0) \end{array} \right.$$

If we fix ξ_0 , the right-hand part of the system (20) is calculated from (21), and ξ_{c0} , ξ_{ci} and ξ_{si} are determined uniquely. The function thus defined (19) would be a solution of the differential equations (17) in the type (18) if the relation (22) is fulfilled. This dependence determines the initial condition of the desired T–periodic solution within the preselected poly-harmonic approximation.

$$(22)\quad \int_0^T \left[\xi(\xi_0,t),t \right] dt = 0$$

By introducing a small artificial parameter $\varepsilon = 1$ in the right part of (17), the provision of the preselected precision is evaluated using the function

$$\chi(\eta - t) \equiv \frac{\sin\frac{(2n+1)(\eta - t)}{2}}{\sin\frac{\eta - t}{2}}$$

where

$$(23)\quad \begin{array}{l} f(x) = \frac{1}{T}\int_0^T f\left[x(\eta),\eta\right]\chi(\eta - t)d\eta - \\ -\varepsilon\left\{ \frac{1}{T}\int_0^T f\left[x(\eta),\eta\right]\chi(\eta - t) - f(x,t)d\eta \right\} \end{array}.$$

Following the basic idea of the method of the small parameter [1], [9], the solution of (17) is expressed by the following power series

$$(24)\quad x(t) = x^{(0)}(t) + \varepsilon x^{(1)}(t) + \varepsilon^2 x^{(2)}(t).....$$

The periodic solution of the infant system

$$(25)\quad \frac{dx^{(0)}}{dt} = \frac{1}{T}\int_0^T f\left[x^{(0)}(\eta)\eta \right]\chi(\eta - t)d\eta ,$$

is obtained in the manner mentioned above.

The differential equation in the first approximation has the form

$$(26)\quad \begin{array}{l} \frac{dx^{(1)}}{dt} = \frac{1}{T}\int_0^T x^{(1)}\frac{\partial f}{\partial x}\left(x^{(0)},\eta\right)\chi(\eta - t)d\eta + \\ + f\left(x^{(0)},t\right) - \frac{1}{T}\int_0^T f\left(x^{(0)},\eta\right)\chi(\eta - t)d\eta, \end{array}$$

which makes it possible to estimate the degree of approximation of the desired solution of (17).

Dependency $x^{(1)}(t) = \tilde{x}^{(1)}(t) + \underset{\sim}{x}^{(1)}(t)$ was $\tilde{x}^{(1)}$ the superposition of the first **n** harmonics, and the higher harmonics of the order of magnitude greater than **n** allows separate operation (26).

The degree of satisfaction with inequality

$$\left| \frac{f\left(x^{(0)},t\right) - \frac{1}{T}\int_0^T f\left(x^{(0)},t\right)\chi(\eta - t)d\eta}{f\left(x^{(0)}t\right)} \right| \ll 1,$$

The magnitude of the correction given by the higher harmonics of the solutions in the first approximation or the authenticity of the priori has chosen poly-harmonic approximation gives information on the magnitude of the correction given by the higher harmonics of the solutions in the first approximation.

IV. NUMERICAL EXAMPLE

One of the periodic solutions of the equation:

$$(27)\quad \ddot{x} + x^3 = 0,2\cos t .$$

Was found in [10].

The non-linear equation (22) defining the initial condition $\left(\xi_0, \frac{d\xi_0}{dt}\right)$ in this concrete case is as follows:

$$(28)\quad \left| \begin{array}{l} g_1\left(\xi_0,\frac{d\xi_0}{dt}\right) = \int_0^{2\pi}\left(-x^3 + 0,2\cos t\right)dt = 0 \\ g_2\left(\xi_0,\frac{d\xi_0}{dt}\right) = \frac{d\xi_0}{dt} - \int_0^{2\pi} t\left(-x^3 + 0,2\cos t\right)dt = 0 \end{array} \right.$$

The solution of the equations (28) with an accuracy of 5.10^{-5} , if **n** is equal to 3, is the point with coordinates $\xi_0 = -0,2069$, $\frac{d\xi_0}{dt} = 0,0000$ for which the harmonic series of Fourier has the form

$$x(t) = -0,2067\cos t - 0,0002\cos 3t .$$

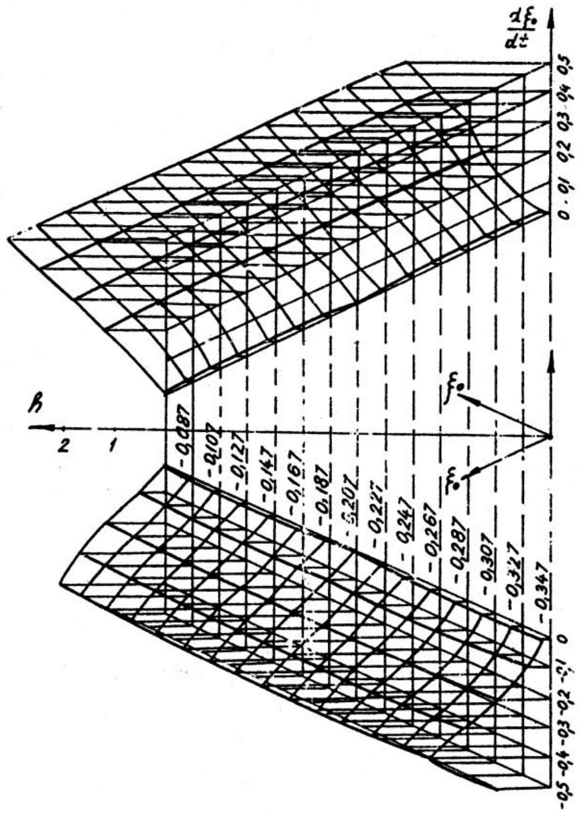

Fig.3: Values of h (28), calculated in the neighbourhood of the initial point

The values $h = \sqrt{g_1^2 + g_2^2}$ calculated in the neighbourhood of the initial point are depicted in Fig.3.

V. CONCLUSION

The difficulties for obtaining approximate solutions of such differential equations are well known when the classical method (Liapunov – Linsted, asymptotic, etc.) of the small parameter is applied. It is necessary to solve these equations by approximations of a higher order.

The advantages of the proposed algorithm for obtaining the approximate T-periodic solutions (steady-state mode of motion) of mechanical systems with one and the finite number of degrees of freedom is that the desired solution in its initial approximation is obtained in poly-harmonic form using the criteria for the extreme number of the harmonics.

The proposed approach to obtaining T-periodic solutions of multidimensional mechanical systems leads to a problem searching for an extreme of a vector-function (22).

ACKNOWLEDGEMENT

The work presented is carried out and financed under the project "University complex for research and development of innovations and knowledge transfer in the field of micro/nanotechnologies and materials, power effectiveness and virtual engineering," contract DUNK-01/3 between National Science Fund, MES of RB and R&D Sector of TU-Sofia.

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