Maximization of Technical Efficiency of a Normal-Truncated Skewed Laplace Stochastic Production Frontier Model

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Abstract— Stochastic Frontier Analysis plays a vital role in the field of mathematical production modelling which is relevant to several sectors like agriculture, health, banking, finance, education, etc. This paper focuses on the derivation of a stochastic production frontier model using Normal distribution and Truncated Skewed- Laplace distribution, namely Normal-Truncated Skewed Laplace Model and the maximization of technical efficiency. This is a generalized form of the Normal-Exponential model. We compute the technical efficiency of a Normal Truncated Skewed Laplace Stochastic Production Frontier Model. The parameters were evaluated using Maximum Likelihood estimation and the maximization of technical efficiency was examined using the second order partial derivatives estimated with respect to the parameters.

Keywords— Technical Efficiency, Normal Truncated Skewed Laplace Distribution, Maximization.

Introduction

Stochastic Frontier Analysis is a mathematical modelling technique widely used in the estimation of efficiency scores. The Stochastic Frontier Analysis was first proposed by Aigner et al (1977) [1] and Meeusen and Van de Broeck(1977)[2]. The purpose of Stochastic Frontier Analysis is to measure how efficient a producer is with the given observations of input and output by using two error terms, u and v. This method is often found to be useful in estimating the values of function in production. Cobb and Douglas (1928) [3], Arrow et al (1961)[4], Berndt and Christensen(1973)[5] and their followers considered the production function to be more flexible in that it needs to take into account some random noise that can affect the production process

Kumbhakar and Lovell (2000)[6] described Production Frontier as "minimum input bundles required to produce various outputs, or the maximum output producible with various input bundles, and a given technology". Producers operating on their production frontier are said to be technically efficient, and the producers operating beneath their production frontier are said to be technically inefficient. The parameters were estimated on the basis of the Aigner and Chu (1986)[7], Afrait (1972)[8], and Richmond(1974)[9]. The Technical Efficiency of a producer is given by $TE_i = \frac{y_i}{f(x_i,\beta)exp\{v_i\}}$ which defines Technical Efficiency as the ratio of observed output to the maximum feasible output, conditional on $exp\{v_i\}$. Technical Efficiency can be attained by the exponential conditional expectation of u given the composed error term ϵ, which is given by $TE_i = exp\left[-E\left({}^{u_i}/_{\epsilon_i}\right)\right]$

The paper focuses on four major sections namely

Section I: Estimation of Normal-Truncated Skewed Laplace Stochastic Production Frontier Model(NTSLSPFM)

Section II: Measurement of Technical Efficiency of Normal-Truncated Skewed Laplace Stochastic Production Frontier Model.

Section III: Estimation of Parameters Using Method of Maximum Likelihood Estimation.

Section IV: Maximization of Technical Efficiency of a Normal -Truncated Skewed Laplace Distribution

I: Estimation of Normal-Truncated Skewed Laplace Stochastic Production Frontier Model(NTSLSPFM).

In the process of estimation of Stochastic Production Frontier Model, the following assumptions were made

i. $v_i \sim iid\ N(0, {\sigma_v}^2)$
ii. $u_i \sim iid$ Truncated Skewed- Laplace Distribution
iii. u_i and v_i are distributed independently of each other and of the regressors.

The probability density function of v is given by

$$f(v) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{-v^2}{2\sigma^2}\right\}, -\infty < v < \infty \quad (1)$$

The probability density function of u is given by

$$f(u) = \frac{(1+\lambda)}{\phi(2\lambda+1)}\left\{2\ exp\left(\frac{-u}{\phi}\right) - exp\left[\frac{-(1+\lambda)u}{\phi}\right]\right\} \quad \text{where } u > 0, \phi > 0, \lambda > 0 \quad (2)$$

Since u and v are independent

$$f(u,v) = f(u)f(v)$$

$$f(u,v) = \frac{1}{\sigma\sqrt{2\pi}} exp\left\{\frac{-v^2}{2\sigma^2}\right\} . \frac{(1+\lambda)}{\phi(2\lambda+1)}\left\{2\ exp\left(\frac{-u}{\phi}\right) - exp\left[\frac{-(1+\lambda)u}{\phi}\right]\right\} \quad (3)$$

$$f(u,v) = \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\left\{2\ exp\left(-\frac{v^2}{2\sigma^2} - \frac{u}{\phi}\right) - exp\left[-\frac{v^2}{2\sigma^2} - \frac{(1+\lambda)u}{\phi}\right]\right\} \quad (4)$$

Substituting $v = u+\in$

$$f(u,v) = \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\left\{2\ exp\left(-\frac{(u+\in)^2}{2\sigma^2} - \frac{u}{\phi}\right) - exp\left[-\frac{(u+\in)^2}{2\sigma^2} - \frac{(1+\lambda)u}{\phi}\right]\right\} \quad (5)$$

The marginal density function of ϵ is

$$f(\in) = \int_0^\infty f(u,\in)du$$

$$f(\in) = \int_0^\infty \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\left\{2\ exp\left(-\frac{(u+\in)^2}{2\sigma^2} - \frac{u}{\phi}\right) - exp\left[-\frac{(u+\in)^2}{2\sigma^2} - \frac{(1+\lambda)u}{\phi}\right]\right\} du \quad (6)$$

$$f(\in) = \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left(-\frac{(u+\in)^2}{2\sigma^2} - \frac{u}{\phi}\right) du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left[-\frac{(u+\in)^2}{2\sigma^2} - \frac{(1+\lambda)u}{\phi}\right] du \quad (7)$$

$$f(\in) = \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{\epsilon^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{u}{\phi}\right] du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{\epsilon^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{(1+\lambda)u}{\phi}\right] du \quad (8)$$

$$f(\in) = \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{u}{\phi}\right] exp\left(-\frac{\epsilon^2}{2\sigma^2}\right) du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{(1+\lambda)u}{\phi}\right] exp\left(-\frac{\epsilon^2}{2\sigma^2}\right) du \quad (9)$$

$$f(\in) = \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{u}{\phi}\right] du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\left(\frac{u^2}{2\sigma^2} + \frac{u\epsilon}{\sigma^2}\right) - \frac{(1+\lambda)u}{\phi}\right] du \quad (10)$$

$$f(\in) = \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u^2}{2\sigma^2} + \frac{2u\epsilon}{\sigma^2} + \frac{2u}{\phi}\right)\right] du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u^2}{2\sigma^2} + \frac{2u\epsilon}{\sigma^2} + \frac{2(1+\lambda)u}{\phi}\right)\right] du \quad (11)$$

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u^2}{2\sigma^2}+\frac{2u\epsilon}{\sigma^2}+\frac{2u}{\phi}+\frac{\in^2}{\sigma^2}-\frac{\in^2}{\sigma^2}+\frac{\sigma^2}{\phi^2}-\frac{\sigma^2}{\phi^2}+\frac{2\in}{\phi}-\frac{2\in}{\phi}\right)\right]du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u^2}{2\sigma^2}+\frac{2u\epsilon}{\sigma^2}+\frac{2(1+\lambda)u}{\phi}+\frac{\in^2}{\sigma^2}-\frac{\in^2}{\sigma^2}+\frac{\sigma^2}{\phi^2}-\frac{\sigma^2}{\phi^2}+\frac{2(1+\lambda)\in}{\phi}-\frac{2(1+\lambda)\in}{\phi}\right)\right]du \quad (12)$$

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]exp\left[-\frac{1}{2}\left(-\frac{\in^2}{\sigma^2}-\frac{\sigma^2}{\phi^2}-\frac{2\in}{\phi}\right)\right]du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]exp\left[-\frac{1}{2}\left(-\frac{\in^2}{\sigma^2}-\frac{\sigma^2}{\phi^2}-\frac{2(1+\lambda)\in}{\phi}\right)\right]du \quad (13)$$

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)exp\left[-\frac{1}{2}\left(-\frac{\in^2}{\sigma^2}-\frac{\sigma^2}{\phi^2}-\frac{2\in}{\phi}\right)\right]\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)exp\left[-\frac{1}{2}\left(-\frac{\in^2}{\sigma^2}-\frac{\sigma^2}{\phi^2}-\frac{2(1+\lambda)\in}{\phi}\right)\right]\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]du \quad (14)$$

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}+\frac{\in^2}{2\sigma^2}+\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(-\frac{\epsilon^2}{2\sigma^2}+\frac{\in^2}{2\sigma^2}+\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]du \quad (15)$$

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]du - \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\int_0^\infty exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]du \quad (16)$$

Let $z=\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)$ $\quad and \;\; dz=-\frac{1}{\sigma}du$

Limits:

u	0	∞
z	$\frac{\in}{\sigma}+\frac{\sigma}{\phi}$	∞

And $z'=\left(\frac{u}{\sigma}+\frac{\in}{\sigma}+\frac{(1+\lambda)\sigma}{\phi}\right)$

$dz'=-\frac{1}{\sigma}du$

Limits:

u	0	∞
z'	$\frac{\in}{\sigma}+\frac{(1+\lambda)\sigma}{\phi}$	∞

$$f(\in)=\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\int_{\frac{\in}{\sigma}+\frac{\sigma}{\phi}}^{\infty}exp\left(-\frac{z^2}{2}\right)\sigma dz-\frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\int_{\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}}^{\infty}exp\left(-\frac{z'^2}{2}\right)\sigma dz' \quad (17)$$

$$f(\in)=\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\Phi\left(\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \quad (18)$$

TO FIND MEAN AND VARIANCE

$$E(\epsilon)=E(v-u)=E(v)-E(u)=0-E(u) \quad (19)$$

$$E(\epsilon)=-E(u) \quad (20)$$

$$E(u)=\int_0^{\infty}uf(u)du \quad (21)$$

$$E(u)=\int_0^{\infty}u\frac{(1+\lambda)}{\phi(2\lambda+1)}\left\{2\,exp\left(\frac{-u}{\phi}\right)-exp\left[\frac{-(1+\lambda)u}{\phi}\right]\right\}du \quad (22)$$

$$E(u)=\frac{(1+\lambda)}{\phi(2\lambda+1)}\left[\int_0^{\infty}2u\,exp\left(\frac{-u}{\phi}\right)du-\int_0^{\infty}u\,exp\left[\frac{-(1+\lambda)u}{\phi}\right]du\right] \quad (23)$$

$$E(u)=\frac{(1+\lambda)}{\phi(2\lambda+1)}\left[2\phi^2\,\Gamma(2)-\frac{\phi^2}{(1+\lambda)^2}\;\Gamma(2)\right] \quad (24)$$

Using kth moment of X-gamma function $E(u)=\frac{\phi^{k+1}(1+\lambda)\Gamma(k+1)}{(2\lambda+1)}\left[2-\frac{1}{(1+\lambda)^{k+1}}\right]$

$$E(u)=\frac{\phi^{k+1}(1+\lambda)\Gamma(k+1)}{(2\lambda+1)}\left[2-\frac{1}{(1+\lambda)^{k+1}}\right] \quad (25)$$

$$E(u)=\frac{(1+\lambda)}{\phi(2\lambda+1)}\left[2\phi^2-\frac{\phi^2}{(1+\lambda)^2}\right] \qquad \text{(since } \Gamma(2)=1\text{)} \quad (26)$$

$$E(u)=\frac{(1+\lambda)}{\phi(2\lambda+1)}\left[\frac{2\phi^2(1+\lambda)^2-\phi^2}{(1+\lambda)^2}\right] \quad (27)$$

$$E(u)=\frac{1}{\phi(2\lambda+1)}\left[\frac{2\phi^2(1+\lambda)^2-\phi^2}{(1+\lambda)}\right] \quad (28)$$

$$E(u)=\frac{\phi^2}{\phi(2\lambda+1)}\left[\frac{2(1+\lambda)^2-1}{(1+\lambda)}\right] \quad (29)$$

$$E(u) = \frac{\phi}{(2\lambda+1)}\left[\frac{2(1+\lambda)^2-1}{(1+\lambda)}\right] \quad (30)$$

$$E(u) = \frac{\phi}{(2\lambda+1)}\left[\frac{2+2\lambda^2-4\lambda-1}{(1+\lambda)}\right] \quad (31)$$

$$E(u) = \phi\left[\frac{1+2\lambda^2-4\lambda-1}{(1+\lambda)(2\lambda+1)}\right] \quad (32)$$

$$E(u) = \phi\left[\frac{1+2\lambda^2-4\lambda-1}{(1+\lambda)(2\lambda+1)}\right] \quad (33)$$

$$E(\epsilon) = -\phi\left[\frac{1+2\lambda^2-4\lambda-1}{(1+\lambda)(2\lambda+1)}\right] \qquad (\text{ since} E(\epsilon) = -E(u)\ ) \quad (34)$$

$$E(u^2) = \int_0^\infty u^2 f(u) du \quad (35)$$

$$E(u^2) = \int_0^\infty u^2 \frac{(1+\lambda)}{\phi(2\lambda+1)}\left\{2\, exp\left(\frac{-u}{\phi}\right) - exp\left[\frac{-(1+\lambda)u}{\phi}\right]\right\} du \quad (36)$$

$$E(u^2) = \frac{(1+\lambda)}{\phi(2\lambda+1)}\left[\int_0^\infty 2u^2\, exp\left(\frac{-u}{\phi}\right) du - \int_0^\infty u^2\, exp\left[\frac{-(1+\lambda)u}{\phi}\right] du\right] \quad (37)$$

$$E(u^2) = \frac{(1+\lambda)}{\phi(2\lambda+1)}\left[2\phi^3\,\Gamma(3) - \frac{\phi^3}{(1+\lambda)^3}\ \Gamma(3)\right] \quad (38)$$

$$E(u^2) = \frac{(1+\lambda)}{\phi(2\lambda+1)}\left[4\phi^3\ - \frac{2\phi^3}{(1+\lambda)^3}\right] \qquad \text{since}\Gamma(3) = 2) \quad (39)$$

$$E(u^2) = \frac{(1+\lambda)}{\phi(2\lambda+1)}\left[\frac{4\phi^3(1+\lambda)^3-2\phi^3}{(1+\lambda)^3}\right] \quad (40)$$

$$E(u^2) = \frac{2\phi^3(1+\lambda)}{\phi(2\lambda+1)}\left[\frac{2(1+\lambda)^3-1}{(1+\lambda)^2}\right] \quad (41)$$

$$E(u^2) = \frac{2\phi^2}{(2\lambda+1)}\left[\frac{2(1+\lambda)^3-1}{(1+\lambda)^2}\right] \quad (42)$$

$$Var\ (u) = E(u^2) - \left(E(u)\right)^2 \quad (43)$$

$$Var\ (u) = \frac{2\phi^2}{(2\lambda+1)}\left[\frac{2(1+\lambda)^3-1}{(1+\lambda)^2}\right] - \left\{\phi\left[\frac{1+2\lambda^2-4\lambda-1}{(1+\lambda)(2\lambda+1)}\right]\right\}^2 \quad (44)$$

$$Var\ (u) = \frac{2\phi^2}{(2\lambda+1)}\left[\frac{2(1+\lambda)^3-1}{(1+\lambda)^2}\right] - \frac{\phi^2\left[2(1+\lambda)^2-1\right]^2}{(1+\lambda)^2(2\lambda+1)^2} \quad (45)$$

$$Var\ (u) = \frac{2\phi^2(2\lambda+1)\left[2(1+\lambda)^3-1\right]-\phi^2\left[2(1+\lambda)^2-1\right]^2}{(1+\lambda)^2(2\lambda+1)^2} \quad (46)$$

$$Var\ (u) = \frac{4\phi^2(2\lambda+1)(1+\lambda)^3}{(1+\lambda)^2(2\lambda+1)^2} - \frac{2\phi^2(2\lambda+1)}{(1+\lambda)^2(2\lambda+1)^2} - \frac{\phi^2\left[2(1+\lambda)^2-1\right]^2}{(1+\lambda)^2(2\lambda+1)^2} \quad (47)$$

$$Var\ (u) = \frac{4\phi^2(1+\lambda)}{(2\lambda+1)} - \frac{2\phi^2}{(1+\lambda)^2(2\lambda+1)} - \frac{\phi^2\left[4(1+\lambda)^4-4(1+\lambda)^2+1\right]}{(1+\lambda)^2(2\lambda+1)^2} \quad (48)$$

$$Var\ (u) = \frac{4\phi^2(1+\lambda)}{(2\lambda+1)} - \frac{2\phi^2}{(1+\lambda)^2(2\lambda+1)} - \frac{4\phi^2(1+\lambda)^4}{(1+\lambda)^2(2\lambda+1)^2} - \frac{\phi^2}{(1+\lambda)^2(2\lambda+1)^2} + \frac{4\phi^2(1+\lambda)^2}{(1+\lambda)^2(2\lambda+1)^2} \quad (49)$$

$$Var\,(u) = \frac{4\phi^2(1+\lambda)}{(2\lambda+1)} - \frac{2\phi^2}{(1+\lambda)^2(2\lambda+1)} - \frac{4\phi^2(1+\lambda)^2}{(2\lambda+1)^2} - \frac{\phi^2}{(1+\lambda)^2(2\lambda+1)^2} + \frac{4\phi^2}{(2\lambda+1)^2} \tag{50}$$

$$Var(\epsilon) = {\sigma_v}^2 + \frac{4\phi^2(1+\lambda)}{(2\lambda+1)} - \frac{2\phi^2}{(1+\lambda)^2(2\lambda+1)} - \frac{4\phi^2(1+\lambda)^2}{(2\lambda+1)^2} - \frac{\phi^2}{(1+\lambda)^2(2\lambda+1)^2} + \frac{4\phi^2}{(2\lambda+1)^2} \tag{51}$$

II: MEASUREMENT OF TECHNICAL EFFICIENCY OF NORMAL-TRUNCATED SKEWED LAPLACE STOCHASTIC PRODUCTION FRONTIER MODEL

Technical Efficiency $TE = exp[-E(u/\epsilon)]$

$$f(u/\epsilon) = \frac{f(u,\epsilon)}{f(\epsilon)} \tag{52}$$

$$f(u/\epsilon) = \frac{\frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\left\{2\,exp\left(-\frac{(u+\epsilon)^2}{2\sigma^2}-\frac{u}{\phi}\right)-exp\left[-\frac{(u+\epsilon)^2}{2\sigma^2}-\frac{(1+\lambda)u}{\phi}\right]\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \tag{53}$$

$$f(u/\epsilon) = \frac{\frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}\left\{\begin{array}{c}2exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]- \\ exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]\end{array}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \tag{54}$$

$$f(u/\epsilon) = \frac{\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]-\frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \tag{55}$$

$$E(u/\epsilon) = \int_0^\infty u\,\frac{\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]-\frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)exp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}du \tag{56}$$

$$E(u/\epsilon) = \frac{\left\{\begin{array}{c}\frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\int_0^\infty uexp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)^2\right]du- \\ \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\int_0^\infty uexp\left[-\frac{1}{2}\left(\frac{u}{\sigma}+\frac{\epsilon}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right]du\end{array}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \tag{57}$$

Let $t = \left(\frac{u}{\sigma} + \frac{\epsilon}{\sigma} + \frac{\sigma}{\phi}\right)$

$dt = \frac{du}{\sigma}$

$u = \sigma(t - \frac{\in}{\sigma} - \frac{\sigma}{\phi})$

Limits

u	t
0	$\frac{\in}{\sigma} + \frac{\sigma}{\phi}$
∞	∞

Let $z = \left(\frac{u}{\sigma} + \frac{\in}{\sigma} + \frac{\sigma(1+\lambda)}{\phi}\right)$

$dz = \frac{du}{\sigma} \quad u = \sigma(t - \frac{\in}{\sigma} - \frac{\sigma(1+\lambda)}{\phi})$

Limits

u	0	∞
Z	$\frac{\in}{\sigma} + \frac{\sigma(1+\lambda)}{\phi}$	∞

$$E(u/\epsilon) = \frac{\left\{\begin{array}{c} \frac{2(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\in}{\phi}\right) \int_{\frac{\in}{\sigma}+\frac{\sigma}{\phi}}^{\infty} \sigma(t - \frac{\in}{\sigma} - \frac{\sigma}{\phi})\, exp\left[-\frac{t^2}{2}\right]\sigma dt - \\ \frac{(1+\lambda)}{\sigma\sqrt{2\pi}\phi(2\lambda+1)} exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\in}{\phi}\right) \int_{\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}}^{\infty} \left(\frac{u}{\sigma} + \frac{\in}{\sigma} + \frac{\sigma(1+\lambda)}{\phi}\right) exp\left[-\frac{z^2}{2}\right]\sigma dz \end{array}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)} exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma} - \frac{\sigma}{\phi}\right) - \frac{(1+\lambda)}{\phi(2\lambda+1)} exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\in}{\phi}\right)\Phi\left(=\frac{\in}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)} \tag{58}$$

$$E(u/\epsilon) = \frac{\left\{\begin{array}{c} \frac{2\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)} exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\in}{\phi}\right)\left[\int_{\frac{\in}{\sigma}+\frac{\sigma}{\phi}}^{\infty} t\ exp\left[-\frac{t^2}{2}\right]dt - (\frac{\in}{\sigma} + \frac{\sigma}{\phi}) \int_{\frac{\in}{\sigma}+\frac{\sigma}{\phi}}^{\infty} exp\left[-\frac{t^2}{2}\right]dt\right] - \\ \frac{\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)} exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\in}{\phi}\right) \\ \left[\int_{\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}}^{\infty} z\, exp\left[-\frac{z^2}{2}\right]dz - \int_{\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}}^{\infty} \left(\frac{\in}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) exp\left[-\frac{z^2}{2}\right]dz\right] \end{array}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)} exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma} - \frac{\sigma}{\phi}\right) - \frac{(1+\lambda)}{\phi(2\lambda+1)} exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}$$

(59)

Let $u = \frac{t^2}{2}$

$du = tdt$

Limits

u	$(\frac{\in}{\sigma}+\frac{\sigma}{\phi})$	∞
t	$(\frac{\in}{\sigma}+\frac{\sigma}{\phi})^2$	∞

Let u=$\frac{z^2}{2}$

$du = zdz$

Limits

z	$(\frac{\in}{\sigma}+\frac{\sigma}{\phi})$	∞
u	$(\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi})^2$	∞

$$E(u/\epsilon)=\frac{\left\{\begin{matrix}\frac{2\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\left[\int_{\left(\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2}^{\infty}\exp(-u)du-(\frac{\in}{\sigma}+\frac{\sigma}{\phi})\int_{\frac{\in}{\sigma}+\frac{\sigma}{\phi}}^{\infty}exp\left[-\frac{t^2}{2}\right]dt\right]-\\ \frac{\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\\ \left[\int_{\left(\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2}^{\infty}\exp(-u)du-\int_{\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}}^{\infty}\left(\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)exp\left[-\frac{z^2}{2}\right]dz\right]\end{matrix}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \quad (60)$$

$$E(u/\epsilon)=\frac{\left\{\begin{matrix}\frac{2\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)exp\left\{-\left(\frac{\in}{\sigma}+\frac{\sigma}{\phi}\right)^2\right\}+\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)(\frac{\in}{\sigma}+\frac{\sigma}{\phi})\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma}{\phi}\right)-\\ \frac{\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)exp\left\{-\left(\frac{\in}{\sigma}+\frac{\sigma(1+\lambda)}{\phi}\right)^2\right\}+\frac{\sigma(1+\lambda)}{\sqrt{2\pi}\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\left(-\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\\ \Phi\left(\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\end{matrix}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\in}{\phi}\right)\Phi\left(-\frac{\in}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \quad (61)$$

$$E(u/\epsilon)=\frac{\left\{\begin{array}{c}\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)+\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\left(\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\\ \frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)+\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\\ \Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\end{array}\right\}}{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \quad (62)$$

$$E(u/\epsilon)=\frac{\left\{\begin{array}{c}2exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)+2exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\left(\frac{\epsilon}{\sigma}+\frac{\sigma}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\\ exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)+exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\\ \Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\end{array}\right\}}{2exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)} \quad (63)$$

III: ESTIMATION OF PARAMETERS USING METHOD OF MAXIMUM LIKELIHOOD ESTIMATION

$$L(sample)=\prod_{i=1}^{N} f(\epsilon_i) \quad (64)$$

$$L(sample)=\prod_{i=1}^{N}\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)-\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \quad (65)$$

The log likelihood function for sample of N producers is

$$lnL=ln\left\{\frac{2(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)\right\}-ln\left\{\frac{(1+\lambda)}{\phi(2\lambda+1)}exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)\ \Phi\left(\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right\} \quad (66)$$

$$lnL=\left\{\left(\frac{\sigma^2}{2\phi^2}+\frac{\epsilon}{\phi}\right)+ln\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma}{\phi}\right)\right\}-\left\{\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)\epsilon}{\phi}\right)+ln\Phi\left(-\frac{\epsilon}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right\} \quad (67)$$

$$lnL=\left\{\left(\frac{\sigma^2}{2\phi^2}+\frac{(y_i-x_i\beta)}{\phi}\right)+ln\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma}{\phi}\right)\right\}-\left\{\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2}+\frac{(1+\lambda)(y_i-x_i\beta)}{\phi}\right)+\ ln\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right\} \quad (68)$$

The parameters $\beta,\sigma,\phi,\lambda$ can be estimated by differentiating the above equation partially with respect to the parameters and equating them to zero.

$$\frac{\partial lnL}{\partial\beta}=-x_i\phi+\frac{x_i}{(y_i-x_i\beta)}+\frac{x_i(1+\lambda)}{\phi}+\frac{x_i\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\sigma\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}=0 \quad (69)$$

$$\frac{\partial lnL}{\partial\phi}=-\frac{(y_i-x_i\beta)}{\phi^2}+\frac{2\sigma^2}{\phi^3}+\frac{1}{\phi}+\frac{(y_i-x_i\beta)(1+\lambda)}{\phi^2}+\frac{(1+\lambda)^2\sigma^2}{\phi^3}+\frac{(1+\lambda)\sigma}{\phi^2}\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}=0 \quad (70)$$

$$\frac{\partial lnL}{\partial \sigma} = \frac{\sigma}{\phi^2} + \frac{1}{\sigma} - \frac{1}{\phi} + \frac{(1+\lambda)^2\sigma}{\phi^2} + \frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}\left(\frac{(y_i-x_i\beta)}{\sigma^2} - \frac{(1+\lambda)}{\phi}\right) = 0 \quad (71)$$

$$\frac{\partial lnL}{\partial \lambda} = -\frac{(y_i-x_i\beta)}{\phi} - \frac{(1+\lambda)\sigma^2}{\phi^3} + \frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}\left(-\frac{\sigma}{\phi}\right) = 0 \quad (72)$$

IV: MAXIMIZATION OF TECHNICAL EFFICIENCY OF A NORMAL -TRUNCATED SKEWED LAPLACE DISTRIBUTION

The second order partial derivatives are estimated with respect to the parameters

$\beta, \sigma, \phi, \lambda$ as

$$\frac{\partial^2 lnL}{\partial \beta^2} = \frac{-x_i(-x_i)}{(y_i-x_i\beta)^2} + \frac{x_i}{\sigma}\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\left[\frac{-(-x_i)}{\sigma}\right] - \Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\left[\frac{-(-x_i)}{\sigma}\right]}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (73)$$

$$\frac{\partial^2 lnL}{\partial \beta^2} = \frac{-x_i(-x_i)}{(y_i-x_i\beta)^2} + \frac{x_i}{\sigma}\left[\frac{-(-x_i)}{\sigma}\right]\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) - \Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (74)$$

$$\frac{\partial^2 lnL}{\partial \beta^2} = \frac{-x_i(-x_i)}{(y_i-x_i\beta)^2} + \frac{{x_i}^2}{\sigma^2}\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) - \left[\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (75)$$

$$\frac{\partial^2 lnL}{\partial \phi^2} = -(-2)\frac{(y_i-x_i\beta)}{\phi^3} + \frac{2\sigma^2(-3)}{\phi^4} - \frac{1}{\phi^2} - \frac{(y_i-x_i\beta)(1+\lambda)}{\phi^3} - \frac{(1+\lambda)^2\sigma^2}{\phi^4} +$$

$$(1+\lambda)\sigma\left\{\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)}\frac{(-2)}{\phi^3} + \frac{1}{\phi^2}\left[\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\left(-\frac{1}{\phi^2}\right)[-(1+\lambda)\sigma] - \Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\left(-\frac{1}{\phi^2}\right)[-(1+\lambda)\sigma]}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right]\right\} \quad (76)$$

$$\frac{\partial^2 lnL}{\partial\phi^2} = -(-2)\frac{(y_i-x_i\beta)}{\phi^3} + \frac{2\sigma^2(-3)}{\phi^4} - \frac{1}{\phi^2} - \frac{(y_i-x_i\beta)(1+\lambda)}{\phi^3} - \frac{(1+\lambda)^2\sigma^2}{\phi^4}$$

$$+(1+\lambda)\sigma\left\{\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\frac{(-2)}{\phi^3} + \frac{1}{\phi^2}\left[\frac{\begin{array}{c}\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\left(-\frac{1}{\phi^2}\right)[-(1+\lambda)\sigma] \\ -\left[\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2\left(-\frac{1}{\phi^2}\right)[-(1+\lambda)\sigma]\end{array}}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right]\right\} \quad (77)$$

$$\frac{\partial^2 lnL}{\partial\phi^2} = -(-2)\frac{(y_i-x_i\beta)}{\phi^3} + \frac{2\sigma^2(-3)}{\phi^4} - \frac{1}{\phi^2} - \frac{(y_i-x_i\beta)(1+\lambda)}{\phi^3} - \frac{(1+\lambda)^2\sigma^2}{\phi^4}$$

$$+(1+\lambda)\sigma\left\{\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\frac{(-2)}{\phi^3} + \frac{\left(-\frac{1}{\phi^2}\right)[-(1+\lambda)\sigma]}{\phi^2}\left[\frac{\begin{array}{c}\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \\ -\left[\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2\end{array}}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right]\right\} \quad (78)$$

$$\frac{\partial^2 lnL}{\partial\phi^2} = -(-2)\frac{(y_i-x_i\beta)}{\phi^3} + \frac{2\sigma^2(-3)}{\phi^4} - \frac{1}{\phi^2} - \frac{(y_i-x_i\beta)(1+\lambda)}{\phi^3} - \frac{(1+\lambda)^2\sigma^2}{\phi^4}$$

$$+(1+\lambda)\sigma\left\{\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\frac{(-2)}{\phi^3} + \frac{(1+\lambda)\sigma}{\phi^4}\left[\frac{\begin{array}{c}\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \\ -\left[\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2\end{array}}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right]\right\} \quad (79)$$

$$\frac{\partial^2 lnL}{\partial\sigma^2} = \frac{1}{\phi^2} - \frac{1}{\sigma^2} + \frac{(1+\lambda)^2}{\phi^2}\frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\left[\frac{(y_i-x_i\beta)(-2)}{\sigma^3}\right] + \frac{\left[\frac{(y_i-x_i\beta)}{\sigma^2}-\frac{(1+\lambda)}{\phi}\right]}{\left(\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}$$

$$\left\{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\frac{(y_i-x_i\beta)(-2)}{\sigma^3} - \left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2\frac{(y_i-x_i\beta)(-2)}{\sigma^3}\right\} \quad (80)$$

$$\frac{\partial^2 lnL}{\partial\sigma^2} = \frac{1}{\phi^2} - \frac{1}{\sigma^2} + \frac{(1+\lambda)^2}{\phi^2} + \frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\left[\frac{(y_i-x_i\beta)(-2)}{\sigma^3}\right] + \left[\frac{(y_i-x_i\beta)}{\sigma^2} - \frac{(1+\lambda)}{\phi}\right]\frac{(y_i-x_i\beta)(-2)}{\sigma^3\left(\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}$$

$$\left\{\begin{matrix}\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \\ -\left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2\end{matrix}\right\} \quad (81)$$

$$\frac{\partial^2 lnL}{\partial\sigma^2} = \frac{1}{\phi^2} - \frac{1}{\sigma^2} + \frac{(1+\lambda)^2}{\phi^2} + \frac{\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)}\left[\frac{(y_i-x_i\beta)(-2)}{\sigma^3}\right] - 2\frac{(y_i-x_i\beta)}{\sigma^3}\left[\frac{(y_i-x_i\beta)}{\sigma^2} - \frac{(1+\lambda)}{\phi}\right]$$

$$\left[\frac{\begin{matrix}\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) \\ -\left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2\end{matrix}}{\left(\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}\right] \quad (82)$$

$$\frac{\partial^2 lnL}{\partial\lambda^2} = -\frac{\sigma^2}{\phi^3} + \left(-\frac{\sigma}{\phi}\right)\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\left(-\frac{\sigma}{\phi}\right) - \left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2\left(-\frac{\sigma}{\phi}\right)}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (83)$$

$$\frac{\partial^2 lnL}{\partial\lambda^2} = -\frac{\sigma^2}{\phi^3} + \left(-\frac{\sigma}{\phi}\right)\left(-\frac{\sigma}{\phi}\right)\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) - \left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (84)$$

$$\frac{\partial^2 lnL}{\partial\lambda^2} = -\frac{\sigma^2}{\phi^3} + \left(\frac{\sigma}{\phi}\right)^2\left\{\frac{\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right) - \left(\Phi'\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}{\left[\Phi\left(-\frac{(y_i-x_i\beta)}{\sigma}-\frac{\sigma(1+\lambda)}{\phi}\right)\right]^2}\right\} \quad (85)$$

$$\frac{\partial^2 lnL}{\partial \lambda^2} = \frac{\sigma^2}{\phi^2} + \left\{ -\frac{1}{\phi} + \frac{\Phi\left(-\frac{(y_i - x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\Phi''\left(-\frac{(y_i - x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) - \left(\Phi'\left(-\frac{(y_i - x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right)^2}{\left[\Phi\left(-\frac{(y_i - x_i\beta)}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)\right]^2} \right\} \tag{86}$$

With the second order partial derivatives estimated above, if they are greater than zero then the production is said to be maximum otherwise minimum.

TECHNICAL EFFICIENCY

The Technical Efficiency of a Normal Truncated Skewed Laplace Stochastic Production Frontier Model is given by

$$TE = exp[-E(u/\epsilon)] \tag{87}$$

$$TE = exp\left[- \frac{\left\{ \begin{array}{c} 2exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma}{\phi}\right) + 2exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\epsilon}{\phi}\right)\left(\frac{\epsilon}{\sigma} + \frac{\sigma}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma}{\phi}\right) - \\ exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\epsilon}{\phi}\right)\phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) + exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\epsilon}{\phi}\right)\left(-\frac{\epsilon}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) \\ \Phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right) \end{array} \right\}}{2exp\left(\frac{\sigma^2}{2\phi^2} + \frac{\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma}{\phi}\right) - exp\left(\frac{\sigma^2(1+\lambda)^2}{2\phi^2} + \frac{(1+\lambda)\epsilon}{\phi}\right)\Phi\left(-\frac{\epsilon}{\sigma} - \frac{\sigma(1+\lambda)}{\phi}\right)} \right] \tag{88}$$

CONCLUSION

Thus with the aim of estimating the technical efficiency of the firm and the individual technical efficiency of a producer the Normal-Truncated Skewed Laplace Production Frontier Model has been derived. The technically inefficient firm can be identified and several productive measures can be implemented to make it technically efficient.

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