Operation Approaches on δ-Open Sets

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Abstract—*The concept of an operation-κ on a family of δ-open sets in topological spaces is introduced. Using the operation κ, the concepts of κ-interior, κ-closure, κ-boundary and κ-exterior are studied.*

I. Introduction

In 1937, Stone[5] initiated the concept of Regular closed sets. Following his work, Velicko[6] introduced the family of δ-open sets in 1968, which are stronger than the family of open sets. A subset of a topological space is called δ-open if it is the union of regular open sets. Further, Velicko investigated the characterization of H-Closed spaces in terms of arbitrary filter bases and showed that, the collection τ_δ of all δ-open sets, is a coarser topology on X.

In 1979, Kasahara[2] defined the concept of an operation α on a topological space and discussed the concept of an α-closed graph of a function. Following this, Jankovic[1] developed the concept of α-closed sets and further investigated functions with α-closed graphs in 1983. Later in 1991, Ogata[3] defined γ-open sets and studied the related topological properties of the associated topology τ_γ and τ. Being motivated by the above works, we introduce operation approaches on δ open sets and notions of κ-interior, κ-closure, κ-boundary and κ-exterior in topological spaces Further, we study the properties of these notions.

II. Preliminaries

Definition 2.1 [5]

Let (X, τ) be a topological space. A subset A of is called ***regular open*** if A = int(cl(A)).

Definition 2.2 [6]

A subset A of a topological space (X, τ) is called ***δ-open*** if it is the union of regular open sets.

III. κ-Operation

Definition 3.1

Let (X,τ) be a topological space. An operation $\kappa:\tau_\delta \rightarrow P(X)$ is a mapping from the family of δ-open sets (τ_δ) to the power set of X such that $V \subseteq V^\kappa$ for every $V \in \tau_\delta$. Here V^κ denotes the value of V under κ.

Example 3.2

Let $X=\{a,b,c\}$ and $\tau=\{X,\phi,\{a\},\{b\},\{a,b\},\{a,c\}\}$ and $\tau_\delta=\{X,\phi,\{b\},\{a,c\}\}$ .Then $\kappa:\tau_\delta\rightarrow P(X)$ defined by $A^\kappa=\begin{cases} A & if\ \ a\in A \\ cl(A) & if\ \ a\notin A\end{cases}$ is a κ-operation on (X,τ) as $A\subseteq A^\kappa$, for every $A\in\tau_\delta$.

Definition 3.3

Let $A\subseteq(X,\tau)$. A point $x\in A$ is called a ***κ-interior*** point of A iff there exist a δ-open neighbourhood N of x such that $N^\kappa\subseteq A$ and we denote the set of all such points by $Int_\kappa(A)$.

Therefore

$Int_{\kappa}(A)=\{x\in A/x\in N\in\tau_{\delta} \quad and \quad N^{\kappa}\subseteq A\}\subseteq A$

.

Example 3.4

Let (X,τ) and κ be defined as in Example 3.2. Take $A=\{a,c\}$, then $Int_{\kappa}(A)=A$.

Definition 3.5

Let (X,τ) be a topological space and $\kappa:\tau_{\delta}\to P(X)$ be a κ-operation. A subset A of X is called ***κ-open*** if every point of A is a κ-interior point.

Example 3.6

Let X,τ,κ and A be defined as in Example3.4. Then A is a κ-open set.

Remark 3.7

A is κ-open iff $A=Int_{\kappa}(A)$

Proof

Necessity: Let A be κ-open. In general, $Int_{\kappa}(A)\subseteq A$. So, it suffices to prove $A\subseteq Int_{\kappa}(A)$. Let $x\in A$. Since A is κ-open, there exists a δ-open neighbourhood U containing x such that $U^{\kappa}\subseteq A$. Then by Definition 3.3, $x\in Int_{k}(A)$. Therefore $A\subseteq Int_{\kappa}(A)$. Hence $A=Int_{\kappa}(A)$.

Sufficiency: Let $x\in A=Int_{k}(A)$. Therefore there exists a δ-open neighbourhood U containing x such that $U^{\kappa}\subseteq A$. Since x is arbitrary, this is true for all $x\in A$. Hence A is κ-open.

Definition 3.8

A subset A of a topological space (X,τ) is called ***κ-closed*** if $X-A$ is κ-open.

Example 3.9

Let X,τ,κ and A be defined as in Example 3.4. Then $X-A=\{b\}$ is κ-closed.

Definition 3.10

A point $x\in X$ is called a ***κ-closure point*** of $A\subseteq X$ if $U^{\kappa}\cap A\neq\phi$, for every δ-open neighbourhood U of x. The set of all κ-closure points is called κ-closure of A and is denoted by $Cl_{\kappa}(A)$.

Example 3.11

Let X,τ,κ and A be defined as in Example 3.2. Let $A=\{b,c\}$ then $Cl_{\kappa}(A)=\{a,b,c\}=X$.

Proposition 3.12

If $A\subseteq B$ then

(i) $Int_{\kappa}(A)\subseteq Int_{\kappa}(B)$

(ii) $Cl_{\kappa}(A)\subseteq Cl_{\kappa}(B)$

Proof

Let $A\subseteq B\subseteq X$.

(i) Let $x\in Int_{\kappa}(A)$ then there exists a δ-open neighbourhood U of x such that $U^{\kappa}\subseteq A$. Since $A\subseteq B$, there exists a δ-open neighbourhood U of x such that $U^{\kappa}\subseteq B$. This implies $x\in Int_{\kappa}(B)$. Hence $Int_{\kappa}(A)\subseteq Int_{\kappa}(B)$.

(ii) Let $x\in Cl_{\kappa}(A)$ then there exists a δ-open neighbourhood U of x such that $U^{\kappa}\cap A\neq\varphi$.

Since $A \subseteq B$, there exists a δ-open neighbourhood U of x such that $U^{\kappa} \cap B \neq \varphi$. Therefore $x \in Cl_{\kappa}(B)$. Hence $Cl_{\kappa}(A) \subseteq Cl_{\kappa}(B)$.

Definition 3.13

A κ-operation $\kappa : \tau_{\delta} \to P(X)$ is called κ-***regular*** if for any δ-open neighbourhoods U and V of $x \in X$, there exists a δ-open neighbourhood W of x such that $U^{\kappa} \cap V^{\kappa} \supseteq W^{\kappa}$.

Definition 3.14

A κ-operation $\kappa : \tau_{\delta} \to P(X)$ is called κ ***-open*** if for every open neighbourhood U of $x \in X$, there exists a δ-open set B such that $x \in B$ and $U^{\kappa} \supseteq B$.

Definition 3.15

A topological space (X, τ) is called κ ***-regular*** if for each open neighbourhood U of $x \in X$, there exists a δ-open neighbourhood V of x such that $V^{\kappa} \subseteq U$.

Theorem 3.16

For subsets A,B of a topological space (X, τ), the following properties are true.

(i) $Int_{\kappa}(Int_{\kappa}(A)) \subseteq Int_{\kappa}(A)$

(ii) $Int_{\kappa}(A \cup B) \supseteq Int_{\kappa}(A) \cup Int_{\kappa}(B)$

(iii) $Int_{\kappa}(A \cap B) = Int_{\kappa}(A) \cap Int_{\kappa}(B)$ if κ is κ-regular.

Proof

(i) We know that $Int_{\kappa}(A) \subseteq A$. That implies $Int_{\kappa}(Int_{\kappa}(A)) \subseteq Int_{\kappa}(A)$.

(ii) We Know, $A \subseteq A \cup B$. This implies $Int_{\kappa}(A) \subseteq Int_{\kappa}(A \cup B)$.
Also,
$B \subseteq A \cup B \Rightarrow Int_{\kappa}(B) \subseteq Int_{\kappa}(A \cup B)$. Therefore
$Int_{\kappa}(A) \cup Int_{\kappa}(B) \subseteq Int_{\kappa}(A \cup B)$.

(iii) $Int_{\kappa}(A \cap B) \subseteq Int_{\kappa}(A) \cap Int_{\kappa}(B)$ is obvious. Let $x \in Int_{\kappa}(A) \cap Int_{\kappa}(B)$. This implies $x \in Int_{\kappa}(A)$ and $x \in Int_{\kappa}(B)$ Therefore there exist δ-open neighbourhood U, V of x such that $U^{\kappa} \subseteq A$ and $V^{\kappa} \subseteq B$. This implies $U^{\kappa} \cap V^{\kappa} \subseteq A \cap B$. Since κ is κ-regular ,there exists a δ-open neighbourhood W of x such that $U^{\kappa} \cap V^{\kappa} \supseteq W^{\kappa}$. This implies $W^{\kappa} \subseteq A \cap B$. This proves $x \in Int_{\kappa}(A \cap B)$, Therefore $Int_{\kappa}(A \cap B) = Int_{\kappa}(A) \cap Int_{\kappa}(B)$.

Theorem 3.17

For a subset A of a topological space (X, τ), then the following properties are true.

(i) $Int_{\kappa}(X - A) = X - cl_{\kappa}(A)$

(ii) $Cl_{\kappa}(X - A) = X - Int_{\kappa}(A)$

(iii) $Int_{\kappa}(A) = X - Cl_{\kappa}(X - A)$

Proof

(i) Let $x \in Int_{\kappa}(X - A)$. There exists a δ-open neighbourhood U of x such that $U^{\kappa} \subseteq X - A$. This implies
$U^{\kappa} \cap A = \phi \Rightarrow x \notin Cl_{\kappa}(A)$
$\Rightarrow x \in X - Cl_{\kappa}(A)$ and conversely.

(ii) Suppose if $x \notin Cl_{\kappa}(X - A)$ Then there exists a δ-open neighbourhood U of x

such that $U^{\kappa} \cap (X - A) = \phi$. This implies $U^{\kappa} \subseteq A$ and thus $x \in Int_{\kappa}(A)$. Therefore $x \notin X - Int_{\kappa}(A)$ and conversely.

(iii) Suppose if $x \notin X - cl_{\kappa}(X - A)$ then $x \in Cl_{\kappa}(X - A)$. That implies there exists a δ-open neighbourhood U of x such that $U^{\kappa} \cap (X - A) \neq \phi$. This implies $U^{\kappa} \cap A = \phi \Rightarrow U^{\kappa} \subseteq A$ .Hence $x \notin Int_{\kappa}(A)$ and conversely.

IV. κ-EXTERIOR AND κ-BOUNDARY

Definition 4.1

κ ***-exterior*** of *A*, written as $Ext_{\kappa}(A)$ is defined as $Ext_{\kappa}(A) = Int_{\kappa}(X - A)$.

Definition 4.2

κ ***-boundary*** of *A*, written as $Bd_{\kappa}(A)$ is defined as the set of points which neither belong to κ-interior of *A* nor κ-exterior of *A*.

Theorem 4.3

In any topological space (X, τ), the following conditions are equivalent:

(i) $X - Bd_{\kappa}(A) = Int_{\kappa}(A) \cup Int_{\kappa}(X - A)$

(ii) $Cl_{\kappa}(A) = Int_{\kappa}(A) \cup Bd_{\kappa}(A)$

(iii) $Bd_{\kappa}(A) = Cl_{\kappa}(A) \cap Cl_{\kappa}(X - A)$

$= Cl_{\kappa}(A) - Int_{\kappa}(A)$

Proof

(iii)$\Rightarrow$ (i) $Int_{\kappa}(A) \cup Int_{\kappa}(X - A) =$

$[Int_{\kappa}(A)]^{cc} \cup [Int_{\kappa}(X - A)]^{cc}$

$= [[Int_{\kappa}(A)]^{c} \cap [Int_{\kappa}(X - A)]^{c}]^{c}$

$= (Cl_{\kappa}(X - A) \cap Cl_{\kappa}(A))^{c}$

$= [Bd_{\kappa}(A)]^{c} = X - Bd_{\kappa}(A)$.

(i)$\Rightarrow$ (ii)Wehave $X - Int_{\kappa}(X - A) = Int_{\kappa}(A) \cup Bd_{\kappa}(A)$.

By Theorem 3.17 (i), we obtain (ii).

(ii)$\Rightarrow$ (iii)By(ii), $Bd_{\kappa}(A) = Cl_{\kappa}(A) - Int_{\kappa}(A)$.

$= Cl_{\kappa}(A) \cap (X - Int_{\kappa}(A)) = Cl_{\kappa}(A) \cap Cl_{\kappa}(X - A)$, by Theorem 3.17 (ii).

Remark 4.4

From Theorem 4.3, we get $Bd_{\kappa}(A) = Bd_{\kappa}(X - A)$.

Proof

$Bd_{\kappa}(A) = X - (Int_{\kappa}(X - A) \cup Int_{\kappa}(A))$

Lemma 4.5

For a subset *A* of *X* , we have the following.

(i) *A* is κ-open iff $A \cap Bd_{\kappa}(A) = \phi$.

(ii) *A* is κ-closed iff $Bd_{\kappa}(A) \subseteq A$.

Proof

(i) Let A be κ-open. Then $(X - A)$ is κ-closed. Therefore $Cl_{\kappa}(X - A) = X - A$. Next, $A \cap Bd_{\kappa}(A) = A \cap [Cl_{\kappa}(A) \cap Cl_{\kappa}(X - A)]$ $= A \cap Cl_{\kappa}(A) \cap (X - A) = \phi$. Conversely, let $A \cap Bd_{\kappa}(A) = \phi$,then $A \cap Cl_{\kappa}(A) \cap Cl_{\kappa}(X - A) = \phi$ or

$Cl_{\kappa}(X-A)\subseteq X-A$ which implies $X-A$ is κ-closed and hence A is κ-open.

(ii) Let A be κ-closed. Then $Cl_{\kappa}(A)=A$. Now $Bd_{\kappa}(A)=Cl_{\kappa}(A)\cap Cl_{\kappa}(X-A)\subseteq Cl_{\kappa}(A)=A$. That is $Bd_{\kappa}(A)\subseteq A$. Conversely, let $Bd_{\kappa}(A)\subseteq A$. Then $Bd_{\kappa}(A)\cap(X-A)=\phi$. Since $Bd_{\kappa}(A)=Bd_{\kappa}(X-A)$, we have $Bd_{\kappa}(X-A)\cap(X-A)=\phi$. By (i) $X-A$ is κ-open and hence A is κ-closed.

Theorem 4.6

For any two subsets *A*,*B* of (X,τ), if κ is regular, then

(i) $ext_{\kappa}(A\cup B)=ext_{\kappa}(A)\cap ext_{\kappa}(B)$

(ii) $bd_{\kappa}(A\cup B)=[bd_{\kappa}(A)\cap cl_{\kappa}(X-B)]\cup [bd_{\kappa}(B)\cap cl_{\kappa}(X-A)]$

(iii) $bd_{\kappa}(A\cap B)=[bd_{\kappa}(A)\cap cl_{\kappa}(B)]\cup [bd_{\kappa}(B)\cap cl_{\kappa}(A)]$

Proof

(i) $ext_{\kappa}(A\cup B)=Int_{\kappa}(X-(A\cup B))$

$=Int_{\kappa}((X-A)\cap(X-B))$

$=Int_{\kappa}(X-A)\cap Int_{\kappa}(X-B)$, by (3.3), since κ is regular.

$=ext_{\kappa}(A)\cap ext_{\kappa}(B)$

(ii) Consider

$$\begin{aligned}
bd_{\kappa}(A\cup B) &= cl_{\kappa}(A\cup B)\cap cl_{\kappa}(X-(A\cup B))\\
&=(cl_{\kappa}(A)\cup cl_{\kappa}(B))\cap cl_{\kappa}((X-A)\cap cl_{\kappa}(X-B))\\
&=(cl_{\kappa}(A)\cup cl_{\kappa}(B))\cap [cl_{\kappa}(X-A)\cap cl_{\kappa}(X-B)]\\
&=(cl_{\kappa}(A)\cap cl_{\kappa}(X-A))\cap(cl_{\kappa}(X-B)\cup cl_{\kappa}(B))\cap[cl_{\kappa}(X-A)\cap cl_{\kappa}(X-B)]\\
&=[bd_{\kappa}(A)\cap cl_{\kappa}(X-B)]\cup[bd_{\kappa}(B)\cap cl_{\kappa}(X-A)]
\end{aligned}$$

(iii)

$$\begin{aligned}
bd_{\kappa}(A\cap B) &= cl_{\kappa}(A\cap B)\cap cl_{\kappa}(X-(A\cap B))\\
&=(cl_{\kappa}(A)\cap cl_{\kappa}(B))\cap cl_{\kappa}((X-A)\cup cl_{\kappa}(X-B))\\
&=(cl_{\kappa}(A)\cap cl_{\kappa}(B))\cap [cl_{\kappa}(X-A)\cup cl_{\kappa}(X-B)]\\
&=([cl_{\kappa}(A)\cap cl_{\kappa}(B)]\cap cl_{\kappa}(X-A))\cup([cl_{\kappa}(A)\cap cl_{\kappa}(B)]\cap cl_{\kappa}(X-B)]\\
&=[bd_{\kappa}(A)\cap cl_{\kappa}(B)]\cup[cl_{\kappa}(A)\cap bd_{\kappa}(B)].
\end{aligned}$$

We also note the following:

(i) $ext_{\kappa}(X-ext_{\kappa}(A))=ext_{\kappa}(A)$.

(ii) $ext_{\kappa}(A\cap B)\supseteq ext_{\kappa}(A)\cup ext_{\kappa}(B)$.

Lemma 4.7

(i) $cl_{\kappa}(A-B)\supseteq cl_{\kappa}(A)-cl_{\kappa}(B)$.

(ii) $Int_{\kappa}(A-B)\subseteq Int_{\kappa}(A)-Int_{\kappa}(B)$.

(iii) If A is κ-open, then $A\cap cl_{\kappa}(B)\subseteq cl_{\kappa}(A\cap B)$.

Proof

(i) Let $x\in cl_{\kappa}(A)-cl_{\kappa}(B)$. Then $x\in cl_{\kappa}(A)$ and $x\notin cl_{\kappa}(B)$. Therefore there exists an open neighbourhood U of x such that $U^{\kappa}\cap A\neq\varphi$, $U^{\kappa}\cap B\neq\varphi$. This gives $U^{\kappa}\cap(A-B)\neq\phi$ or $x\in cl_{\kappa}(A-B)$. This proves (i).

(ii) follows easily from (i).

(iii) Since A is κ-open, $A=Int_{\kappa}(A)$.

Now $A\cap cl_{\kappa}(B)=cl_{\kappa}(B)\cap Int_{\kappa}(A)$

$=cl_{\kappa}(B)-(X-Int_{\kappa}(A))$

$=cl_{\kappa}(B)-Cl_{\kappa}(X-A)$

$\subseteq cl_{\kappa}(B-(X-A))$

$= cl_{\kappa}(B \cap A) = cl_{\kappa}(A \cap B)$ or $A \cap cl_{\kappa}(B) \subseteq cl_{\kappa}(A \cap B)$. This completes the proof.

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