COM-Poisson Polya-Aeppli Distribution

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Abstract

COM-Poisson (Conway-Maxwell Poisson) distribution is a generalization of some well known distributions (Poisson, Bernoulli, Geometric) and this distribution is more flexible for over and under dispersed data. In this paper, the COM-Poisson Polya Aeppli distribution (Compound COM-Poisson distribution with geometric compounding distribution) is introduced. Its properties are studied. The parameters are estimated by the method of profile likelihood estimation.

Key Words: COM-Poisson distribution, Polya-Aeppli distribution, COM-Poisson Polya-Aeppli distribution, Profile likelihood estimation.

1 Introduction

The COM-Poisson distribution is a generalization of Poisson distribution. This distribution is not only a generalization of Poisson distribution, it is the generalization of some well known discrete distributions like negative binomial, Bernoulli and Geometric distributions. In 1962, Conway & Maxwell introduced this distribution in the context of queuing systems. In 2005, Galit Shmueli [8] revived this distribution and used for fitting discrete data. They use the acronym COM-Poisson for this distribution. The COM-Poisson consists of an extra parameter, which we denote by ν and which governs the rate of decay of successive ratios of probabilities such that

$$\frac{P(X = x - 1)}{P(X = x)} = \frac{x^\nu}{\lambda} \tag{1}$$

The COM-Poisson application from a theoretical point of view since it belongs to the exponential family as well as to the two-parameter power series distributions family. Its appeals from a practical point of view is even stronger, it is easy to use flexible for fitting over and under-dispersed data.
The probability mass function of Poisson distribution is

$$P(N(t) = k) = \frac{\lambda^k}{k!} e^{-\lambda} \tag{2}$$

One of the most important properties of the Poisson distribution is its equidispersion. (ie) the Poisson distribution variance and mean are equal. Then the corresponding Fisher index, which is defined as the ratio of the variance to the mean is equal to one. In many practical applications, the equidispersion property of the Poisson distribution is not observed in the count data at hand, it motivates the search for more flexible models for this type of data.

The Polya-Aeppli distribution was described by Polya in 1930, he ascribed the derivation of the distribution to his student Aeppli in a Zurich thesis in 1924.

In this paper we consider the COM-Poisson distribution and derive the mean and variance using Probability generating function. In this paper we define the Compound COM-Poisson distribution with geometric compounding distribution and it name it as COM-Poisson Polya-Aeppli ditribuion and its properties are also derived.

This paper is organised as follows: Section 2 describes the study of COM-Poisson distribution.In section 3 defined the COM-Poisson Polya-Aeppli distribution and discussed some of its properties. Section 4 appeals the profile likelihood estimation of the newly introduced distribution. In section 5 applications are given and also concludes this paper.

2 COM-Poisson Distribution

The probability density function of COM-Poisson distribution [8] is

$$P(X = x) = \frac{\lambda^x}{(x!)^\nu} \frac{1}{Z(\lambda, \nu)}, \qquad x = 0, 1, 2, ... \tag{3}$$

where $Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}$ for $\lambda > 0$ and $\nu \geq 0$.
The probability generating function of COM-Poisson distribution is

$$G_X(s) = \frac{Z(\lambda s, \nu)}{Z(\lambda, \nu)} \tag{4}$$

The mean and variance can be calculated from the first and second derivatives of the probability generating function and then setting s=1

$$\begin{aligned}
G_X^{'}(s) &= \frac{d}{ds}[G_X(s)] = \frac{d}{ds}\left(\frac{Z(\lambda s,\nu)}{Z(\lambda,\nu)}\right) \\
&= \frac{1}{Z(\lambda,\nu)}\sum_{j=0}^{\infty}\frac{j\lambda(\lambda s)^{j-1}}{(j!)^{\nu}} \\
G_X^{'}(1) &= \frac{\lambda}{Z(\lambda,\nu)}\sum_{j=0}^{\infty}\frac{j(\lambda)^{j-1}}{(j!)^{\nu}} \\
Mean(X) = G_X^{'}(1) &= \frac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} \\
G_X^{''}(s) = \frac{d^2}{ds^2}[G_X(s)] &= \frac{d^2}{ds^2}\left(\frac{Z(\lambda s,\nu)}{Z(\lambda,\nu)}\right) \\
&= \frac{1}{Z(\lambda,\nu)}\sum_{j=0}^{\infty}\frac{j(j-1)\lambda^2(\lambda s)^{j-2}}{(j!)^{\nu}} \\
G_X^{''}(1) &= \frac{\lambda^2}{Z(\lambda,\nu)}\sum_{j=0}^{\infty}\frac{j(j-1)(\lambda)^{j-2}}{(j!)^{\nu}} = \frac{\lambda^2 Z_{\lambda\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} \\
Var(X) &= G_X^{''}(1) + G_X^{'}(1) - [G_X^{'}(1)]^2 \\
&= \frac{\lambda^2 Z_{\lambda\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} + \frac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} - \left[\frac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)}\right]^2 \\
\text{where}\quad Z_{\lambda}(\lambda,\nu) &\equiv \frac{d}{d\lambda}[Z(\lambda,\nu)], \\
Z_{\lambda\lambda}(\lambda,\nu) &\equiv \frac{d^2}{d\lambda^2}[Z(\lambda,\nu)]
\end{aligned}$$

From these, we find the ratio between variance and mean to be,

$$\begin{aligned}
\frac{Var(X)}{Mean(X)} &= \frac{\dfrac{\lambda^2 Z_{\lambda\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} + \dfrac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} - \left[\dfrac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)}\right]^2}{\dfrac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)}} \\
&= \frac{\lambda Z_{\lambda\lambda}(\lambda,\nu)}{Z_{\lambda}(\lambda,\nu)} - \frac{\lambda Z_{\lambda}(\lambda,\nu)}{Z(\lambda,\nu)} + 1
\end{aligned}$$

3 COM-Poisson Polya-Aeppli distribution

Suppose that the several events can happen simultaneously at an instant, we have a cluster (of occurences) at a point.
Assume that there are Y independent random variables of the form X, and N denotes the sum of these random variables.

(ie)

$$N = X_1 + X_2 + ... + X_Y$$

Then, the COM-Poisson Polya-Aeppli model is derived by the following assumptions

(i) X represents the number of objects within a cluster and X follows geometric distribution with parameter $(1-\rho)$
(ie)
$$X \sim Geo(1-\rho)$$

(ii) Y represents the number of clusters and Y follows COM-Poisson distribution with parameters λ and ν.
(ie)
$$Y \sim CMP(\lambda, \nu)$$

This random variable, N formed by compounding these two random variables X and Y gives the COM-Poisson Polya-Aeppli distribution with parameters λ, ν and ρ. Its probability generating function (PGF) can be derived as follows. First, we have the probability mass function (PMF) of X is

$$P(X = x) = (1-\rho)\rho^{x-1}, \qquad x = 1, 2, ...$$

Its probability generating function is

$$\begin{aligned} G_X(s) = E(s^X) &= \sum_{x=1}^{\infty} s^x P(X = x) \\ &= \sum_{x=1}^{\infty} s^x (1-\rho)\rho^{x-1} \\ &= (1-\rho)s \sum_{x=1}^{\infty} (\rho s)^x \end{aligned}$$

$$G_X(s) = \frac{(1-\rho)s}{1-\rho s} \tag{5}$$

Also, the random variable Y having probability mass function in (3) and the probability generating function is

$$G_Y(s) = \frac{Z(\lambda s, \nu)}{Z(\lambda, \nu)} \tag{6}$$

Since $X_i's$ are iid and independent of Y, the Probability generating function of the random variable N can be derived as follows

$$\begin{aligned} G_N(s) = E(s^N) &= E(s^{X_1+X_2+...+X_Y}) \\ &= \sum_{y=0}^{\infty} E(s^{X_1+X_2+...+X_Y}/Y = y)P(Y = y) \end{aligned} \tag{7}$$

$$
\begin{aligned}
&= \sum_{y=0}^{\infty} [E(s^x)]^y P(Y=y) \\
&= G_Y(G_X(s)) \\
&= \frac{Z(\lambda G_X(s), \nu)}{Z(\lambda, \nu)} \\
&= \frac{1}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{1}{(j!)^\nu} \left[\frac{\lambda s(1-\rho)}{1-\rho s}\right]^j \\
&= \frac{1}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{1}{(j!)^\nu} \left[(\lambda s(1-\rho))^j (1-\rho s)^{-j}\right]
\end{aligned} \tag{8}
$$

Collecting the coefficient of s^n in the above series we get

$$
P(N=n) = \frac{1}{Z(\lambda,\nu)} \sum_{j=1}^{n} \frac{1}{(j!)^\nu} \binom{n-1}{j-1} (\lambda(1-\rho))^j \rho^{n-j} \qquad \text{for} \quad n = 1, 2, \ldots
$$

$\therefore$ The probability mass function of N is

$$
P(N=n) = \begin{cases} \dfrac{1}{Z(\lambda,\nu)} & \text{for} \quad n = 0 \\ \dfrac{1}{Z(\lambda,\nu)} \sum_{j=1}^{n} \dfrac{1}{(j!)^\nu} \binom{n-1}{j-1} (\lambda(1-\rho))^j \rho^{n-j} & \text{for} \quad n = 1, 2, \ldots \end{cases} \tag{9}
$$

The mean and variance are given by

$$
\begin{aligned}
G_N^{'}(s) &= \frac{1}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{j}{(j!)^\nu} \left(\frac{\lambda s(1-\rho)}{1-\rho s}\right)^{j-1} \frac{(1-\rho s)\lambda(1-\rho) - \lambda s(1-\rho)\rho}{(1-\rho s)^2} \\
&= \frac{1}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{j}{(j!)^\nu} \left(\frac{\lambda s(1-\rho)}{1-\rho s}\right)^{j-1} \frac{(1-\rho)}{(1-\rho s)^2} \\
G_N^{'}(1) &= \frac{\lambda}{(1-\rho) Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{j(\lambda)^{j-1}}{(j!)^\nu}
\end{aligned}
$$

$$
Mean(N) = G_N^{'}(1) = \frac{\lambda Z_\lambda(\lambda,\nu)}{(1-\rho) Z(\lambda,\nu)} \tag{10}
$$

$$
\begin{aligned}
G_N^{''}(s) &= \frac{\lambda(1-\rho)}{Z(\lambda,\nu)} \sum_{j=0}^{\infty} \frac{j}{(j!)^\nu} \frac{(1-\rho s)^2 \left(\frac{\lambda s(1-\rho)}{1-\rho s}\right)^{j-2} \frac{\lambda(1-\rho)}{(1-\rho s)^2} - \left(\frac{\lambda s(1-\rho)}{1-\rho s}\right)^{j-1} 2\rho(1-\rho s)}{(1-\rho s)^4} \\
G_N^{''}(1) &= \frac{1}{Z(\lambda,\nu)(1-\rho)^2} [\lambda^2 Z\lambda\lambda(\lambda,\nu) + 2\rho\lambda Z_\lambda(\lambda,\nu)] \\
Var(N) &= G_N^{''}(1) + G_N^{'}(1) - [G_N^{'}(1)]^2
\end{aligned}
$$

$$Var(N) = \frac{1}{Z(\lambda,\nu)(1-\rho)^2}\left[\lambda^2 Z_{\lambda\lambda}(\lambda,\nu) + (1+\rho)\lambda Z_\lambda(\lambda,\nu) - \frac{\lambda^2[Z_\lambda(\lambda,\nu)]^2}{Z(\lambda,\nu)}\right] \tag{11}$$

The ratio between variance and mean is

$$\frac{Var(N)}{Mean(N)} = \frac{\frac{1}{Z(\lambda,\nu)(1-\rho)^2}\left[\lambda^2 Z_{\lambda\lambda}(\lambda,\nu) + (1+\rho)\lambda Z_\lambda(\lambda,\nu) - \frac{\lambda^2[Z_\lambda(\lambda,\nu)]^2}{Z(\lambda,\nu)}\right]}{\frac{\lambda Z_\lambda(\lambda,\nu)}{(1-\rho)Z(\lambda,\nu)}}$$

$$\frac{Var(N)}{Mean(N)} = \frac{\lambda Z_{\lambda\lambda}(\lambda,\nu)}{(1-\rho)Z_\lambda(\lambda,\nu)} - \frac{\lambda Z_\lambda(\lambda,\nu)}{(1-\rho)Z(\lambda,\nu)} + \frac{(1+\rho)}{1-\rho} \tag{12}$$

According to ratio between variance and mean, when the under dispersion and over dispersion occurred details given in the following table

When $\rho \geq 0.5$& $\nu \geq 1$	When $\rho \geq 0.5$& $\nu \leq 1$	When $\rho < 0.5$ & $\nu < 0.5$	When $\rho < 0.5$ & $\nu > 0.5$
Over Dispersion	Over Dispersion	Under Dispersion	Over Dispersion

4 Profile likelihood estimation

The parameters in the mass/density function are estimated using the method of profile likelihood. Based on the observations $N_1, N_2, ..., N_n$ from COM-Poisson Polya-Aeppli distribution with unknown parameters $\lambda > 0$, $\nu \geq 0$ and $0 \leq \rho < 1$ can be obtained by maximizing the log-likelihood function.
The likelihood function of the observed data can be written as

$$\begin{aligned} L &= \prod_{i=1}^{n} P(N = N_i) \\ &= \prod_{i=1}^{n} \frac{1}{Z(\lambda,\nu)} \sum_{j=1}^{N_i} \frac{1}{(j!)^\nu}\binom{N_i - 1}{j-1}(\lambda(1-\rho))^j \rho^{N_i - j} \end{aligned}$$

The log likelihood function is

$$l = logL = -nlog\left[\sum_{j=0}^{\infty}\frac{\lambda^j}{(j!)^\nu}\right] + \sum_{i=1}^{n} log\left[\sum_{j=1}^{N_i}\frac{1}{(j!)^\nu}\binom{N_i-1}{j-1}(\lambda(1-\rho))^j\rho^{N_i-j}\right] \tag{13}$$

Differentiating equation (13) partially with respect to λ and equating to zero

$$\frac{\partial l}{\partial \lambda} = 0$$

$$0 = \sum_{i=1}^{n}\left[\frac{\sum_{j=1}^{N_i}\frac{j\lambda^{j-1}}{(j!)^\nu}\binom{N_i-1}{j-1}(1-\rho)^j\rho^{N_i-j}}{\sum_{j=1}^{N_i}\frac{\lambda^j}{(j!)^\nu}\binom{N_i-1}{j-1}(1-\rho)^j\rho^{N_i-j}}\right] - n\left[\frac{\sum_{j=0}^{\infty}\frac{j\lambda^{j-1}}{(j!)^\nu}}{\sum_{j=0}^{\infty}\frac{\lambda^j}{(j!)^\nu}}\right]$$

Differentiating equation (13) partially with respect to ρ and equating to zero

$$\frac{\partial l}{\partial \rho} = 0$$

$$0 = \sum_{i=1}^{n} \left[\frac{\sum_{j=1}^{N_i} \frac{1}{(j!)^\nu} \binom{N_i-1}{j-1} \left[(\lambda(1-\rho))^j (N_i - j)\rho^{N_i-j-1} - \lambda j \rho^{N_i-j} (\lambda(1-\rho))^{j-1} \right]}{\sum_{j=1}^{N_i} \frac{1}{(j!)^\nu} \binom{N_i-1}{j-1} (\lambda(1-\rho))^j \rho^{N_i-j}} \right]$$

Then $\max\limits_{\lambda,\nu,\rho} log(\lambda,\nu,\rho|n) = \max\limits_{\nu} \left[\max\limits_{\lambda,\rho} log(\lambda,\rho|n) \right]$ is calculated.
The estimators of λ, ν and ρ are estimated numerically using MATLAB.

5 Applications and Conclusion

COM-Poisson Polya-Aeppli distribution is used for bus drivers accidents. Based on the following assumptions the COM-Poisson Polya-Aeppli model is derived.

- Every driver is liable to have 'spells', where the number of spells per driver in a given period of time is geometric with the same parameter ρ for all drivers.
- The performance of a driver during a spell is substandard, he is liable to have a COM-Poisson number of accidents during the spell with the same parameters λ and ν for all drivers.
- Each driver behaves independently.
- No accidents can be occur outside of a spell.

This distribution is also applicable for plant population.
The COM-Poisson Polya-Aeppli distribution is introduced and its properties are derived. Also the parameters of COM-Poisson Polya-Aeppli distribution are estimated.

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