On Quasi ψ^*a-Open Maps in Topological Spaces

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Abstract— *In this paper we introduce the concept of quasi $\psi^*\alpha$-open maps in topological spaces and study some of their basic properties and characterizations.*

Keywords— *Quasi $\psi^*\alpha$-open maps, $\psi^*\alpha$-closed sets, $\psi^*\alpha$-closure and $\psi^*\alpha$-interior.*

I. Introduction

Many different terms of open maps have been introduced over the years. Various interesting problems arise when one consider openness. Its importance is significant in various areas of Mathematics and related sciences. As a generalization of closed sets, Balamani and Parvathi [1] introduced and studied a new class of generalized closed sets called $\psi^*\alpha$-closed sets in topological spaces. In this paper we introduce a new class of maps called quasi $\psi^*\alpha$-open maps and quasi $\psi^*\alpha$-closed maps in topological spaces and obtain their basic properties.

II. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent non-empty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. The interior and closure of a subset A of a space (X, τ) are denoted by int(A) and cl(A) respectively.

Definition 2.1 Let (X, τ) be a topological space. A subset A of (X, τ) is called a

1. Semi open set [6] if $A \subseteq cl(int(A))$ and a semi closed set if $int(cl(A)) \subseteq A$
2. α-open set [8] if $A \subseteq int(cl(int(A)))$ and a α-closed set if $cl(int(cl(A))) \subseteq A$

The intersection of all semi closed (resp. α -closed) subsets of (X, τ) containing A is called the semi closure (resp. α-closure) of A and is denoted scl(A) (resp. αcl(A)).

Definition 2.2 A subset A of a topological space (X, τ) is called

1) generalized closed set (g-closed) [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

2) semi-generalized closed set (sg-closed) [5] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).

3) ψ-closed set [10] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ).

4) ψg-closed set [9] if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

5) $\psi^*\alpha$-closed set [1] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is ψg-open in (X, τ).

6) The $\psi^*\alpha$-closure of A, denoted by $\psi^*\alpha cl(A)$ is defined as the intersection of all $\psi^*\alpha$-closed sets containing A [1]

7) The $\psi^*\alpha$-interior of A, denoted by $\psi^*\alpha int(A)$ is defined as the union of all $\psi^*\alpha$-open sets contained in A

Definition 2.3 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(i) Continuous [7] if $f^{-1}(V)$ is closed in (X, τ) for every closed set V of (Y, σ).

(ii) $\psi^*\alpha$-continuous [2] if $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ) for every closed set V of (Y, σ).

(iii) $\psi^*\alpha$-irresolute [3] if $f^{-1}(V)$ is $\psi^*\alpha$-closed in (X, τ) for every $\psi^*\alpha$-closed set V of (Y, σ).

Definition 2.4

(i) A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\psi^*\alpha$-closed if f(V) is $\psi^*\alpha$-closed in (Y, σ) for each closed set V in (X, τ) [4]

(ii) A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\psi^*\alpha$-open if f(V) is $\psi^*\alpha$ - open in (Y, σ) for each open set V in (X, τ) [4]

III. Quasi $\psi^*\alpha$- Open Maps

Definition 3.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **quasi $\psi^*\alpha$-open** if f(V) is open in (Y, σ) for every $\psi^*\alpha$-open set V in (X, τ).

Example 3.2 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a},{b}, {a, b}, {a, c}, Y}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is quasi $\psi^*\alpha$-open.

Theorem 3.3 A map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-open if and only if for every subset U of (X, τ), $f(\psi^*\alpha int(U)) \subset int(f(U))$.

Proof: Let f be a quasi $\psi^*\alpha$-open map. Let U be a subset of (X, τ). Since int(U) ⊂ U and $\psi^*\alpha int(U)$ is a $\psi^*\alpha$-open set and $\psi^*\alpha int(U)) \subset U$, $f(\psi^*\alpha int(U)) \subset f(U))$. As f is quasi $\psi^*\alpha$-open, $f(\psi^*\alpha int(U))$ is open and $f(\psi^*\alpha int(U)) = intf(\psi^*\alpha int(U)) \subset int(f(U))$. Therefore $f(\psi^*\alpha int(U)) \subset int(f(U))$.

Conversely, assume that U is a $\psi^*\alpha$-open set in (X, τ).Then $f(U) = f(\psi^*\alpha int(U)) \subset int(f(U))$ but $int(f(U)) \subset f(U)$. Consequently, f(U) = int(f(U)) and hence f is quasi $\psi^*\alpha$-open.

Theorem 3.4 If f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-open then $\psi^*\alpha int(f^{-1}(G)) \subset f^{-1}(int(G))$, for every subset G of (Y, σ).

Proof: Let G be any arbitrary subset of (Y, σ). Then $\psi^*\alpha int(f^{-1}(G))$ is a $\psi^*\alpha$-open set in (X, τ). Since f is quasi $\psi^*\alpha$-open, $f(\psi^*\alpha int(f^{-1}(G))) \subset int(f(f^{-1}(G))) \subset int(G)$. Thus $\psi^*\alpha int(f^{-1}(G)) \subset f^{-1}(int(G))$.

Definition 3.5 A subset A is called a **$\psi^*\alpha$-neighbourhood** of a point x of (X, τ) if there exists a $\psi^*\alpha$-open set U such that x ∈ U ⊂ A.

Theorem 3.6 Let f : (X, τ) → (Y, σ) be a map. Then the following statements are equivalent:

(i) f is quasi $\psi^*\alpha$-open map
(ii) For each subset U of (X, τ), $f(\psi^*\alpha int(U)) \subset int(f(U)$
(iii) For each x ∈ X and each $\psi^*\alpha$-neighbourhood U of x in (X, τ), there exists a neighbourhood V of f(x) in (Y, σ) such that V ⊂ f(U).

Proof: (i) ⇒ (ii) It follows from **Theorem 3.3.**

(ii) ⇒ (iii) Let x ∈ X and U be an arbitrary $\psi^*\alpha$-neighbourhood of x in (X, τ). Then there exists a $\psi^*\alpha$-open set V in (X, τ) such that x ∈ V ⊂ U. Then by (ii), $f(V) = f(\psi^*\alpha int(V)) \subset int(f(V))$ and hence f(V) = int(f(V)). Therefore f(V) is open in (Y, σ) such that f(x) ∈ f(V) ⊂ f(U).

(iii) ⇒ (i) Let U be an arbitrary $\psi^*\alpha$-open set in (X, τ). Then for each y ∈ f(U), by (iii) there exists a neighbourhood V_y of y in (Y, σ) such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exists an open set W_y in (Y, σ) such that $y \in W_y \subset V_y$. Thus $f(U) = \cup\{W_y : y \in f(U)\}$ which is an open set in (Y, σ). This implies that f is quasi $\psi^*\alpha$-open.

Theorem 3.7 A map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-open if and only if for any subset B of (Y, σ) and for any $\psi^*\alpha$-closed set F in (X, τ) containing $f^{-1}(B)$, there exists a closed set G of (Y, σ) containing B such that $f^{-1}(G) \subset F$.

Proof: Suppose f is quasi $\psi^*\alpha$-open. Let B ⊂ Y and F be a $\psi^*\alpha$-closed set in (X, τ) containing $f^{-1}(B)$. Let G = Y – [f(X – F)]. It is clear that $f^{-1}(B) \subset F$ which implies B ⊂ G. Since f is quasi $\psi^*\alpha$-open, G is a closed set of (Y, σ) and $f^{-1}(G) \subset F$.

Conversely, let U be a $\psi^*\alpha$-open set in (X, τ) and let B = Y – f(U). Then X – U is a $\psi^*\alpha$-closed set in (X, τ) containing $f^{-1}(B)$. By hypothesis, there exists a closed set F in (Y, σ) such that B ⊂ F and $f^{-1}(F) \subset X - U$. Hence f(U) ⊂ Y – F. Now B ⊂ F implies Y – F ⊂ Y – B = f(U). Thus f(U) = Y – F is open and hence f is a quasi $\psi^*\alpha$-open map.

Theorem 3.8 A map f : (X, τ) → (Y, σ) is quasi $\psi^*\alpha$-open if and only if $f^{-1}(cl(B)) \subset \psi^*\alpha cl(f^{-1}(B))$, for every subset B of (Y,σ).

Proof: Suppose f is quasi $\psi^*\alpha$ -open. For any subset B of (Y, σ), $f^{-1}(B) \subset \psi^*\alpha cl(f^{-1}(B))$. Therefore by Theorem 3.7, there exists a closed set F in (Y, σ) such that B ⊂ F and $f^{-1}(F) \subset \psi^*\alpha\ cl(f^{-1}(B))$. Therefore $f^{-1}(cl(B)) \subset f^{-1}(F) \subset \psi^*\alpha cl(f^{-1}(B))$.

Conversely, let B ⊂ Y and F be $\psi^*\alpha$-closed in (X, τ) containing $f^{-1}(B)$. Let W = cl(B), then B ⊂ W and W is closed and $f^{-1}(W) \subset \psi^*\alpha cl(f^{-1}(B)) \subset F$. Then by Theorem 3.7, f is quasi $\psi^*\alpha$-open.

Proposition 3.9 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z,η) be two maps and g∘f : (X, τ) → (Z,η) is quasi $\psi^*\alpha$-open. If g is continuous and injective, then f is quasi $\psi^*\alpha$-open.

Proof: Let U be a $\psi^*\alpha$-open set in (X, τ). Then (g∘f)(U) is open in (Z, η) as g∘f is quasi $\psi^*\alpha$-open. Since g is an injective continuous map, $g^{-1}((g\circ f)(U)) = f(U)$ is open in (Y, σ). Hence f is quasi $\psi^*\alpha$-open.

Definition 3.10 A map f : (X, τ) → (Y, σ) is called **quasi $\psi^*\alpha$-closed** if f(V) is closed in (Y, σ) for each $\psi^*\alpha$ - closed set V in (X, τ).

Example 3.11 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a}, {b}, {a, b}, {a, c}, Y}.

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is quasi $\psi^*\alpha$-closed.

Proposition 3.12 Every quasi $\psi^*\alpha$-closed map is closed as well as $\psi^*\alpha$-closed but not conversely

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 3.13 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {b}, {a, b}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is clearly $\psi^*\alpha$-closed as well as closed but not quasi $\psi^*\alpha$-closed, since {c} is $\psi^*\alpha$ - closed in (X, τ) but f({c}) = {b} is not closed in (Y, σ).

Proposition 3.14 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z,\eta)$ be any two maps such that $g \circ f : (X, \tau) \rightarrow (Z,\eta)$ is quasi $\psi^*\alpha$-closed. If f is $\psi^*\alpha$-irresolute and surjective, then g is closed.

Proof: Let U be closed in (Y, σ). Since every closed set is $\psi^*\alpha$-closed and f is $\psi^*\alpha$-irresolute,$f^{-1}(U)$ is $\psi^*\alpha$-closed in (X, τ). Since $g \circ f$ is quasi $\psi^*\alpha$-closed and f is surjective, $(g \circ f)(f^{-1}(U)) = g(U)$ is closed in (Z, η). Hence g is closed.

Proposition 3.15 If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z,\eta)$ are quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed) maps then $g \circ f : (X, \tau) \rightarrow (Z,\eta)$ is also quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed).

Proof: Let V be any $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed) set in (X, τ). Then f(V) is open (resp. closed) in (Y, σ) as f is quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed). Since every open (resp. closed) set is $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed), f(V) is $\psi^*\alpha$-open (resp. $\psi^*\alpha$-closed) in (Y, σ). Since g is quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed), $(g \circ f)(V) = g(f(V))$ is open (resp. closed) in (Z, η). Thus $g \circ f$ is quasi $\psi^*\alpha$-open (resp. quasi $\psi^*\alpha$-closed).

Definition 3.16 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **strongly $\psi^*\alpha$-closed** if f(V) is $\psi^*\alpha$-closed in (Y, σ) for each $\psi^*\alpha$-closed set V in (X, τ).

Example 3.17 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is strongly $\psi^*\alpha$-closed.

Proposition 3.18 Every strongly $\psi^*\alpha$-closed map is $\psi^*\alpha$-closed but not conversely

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 3.19 Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, X} and σ = {ϕ, {a, b}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by f(a) = b, f(b) = a, f(c) = c. Then f is clearly $\psi^*\alpha$-closed but not strongly $\psi^*\alpha$-closed, since {b} is $\psi^*\alpha$-closed in (X, τ) but f({b}) = {a} is not $\psi^*\alpha$-closed in (Y, σ).

Proposition 3.20 Every quasi $\psi^*\alpha$-closed map is strongly $\psi^*\alpha$ -closed but not conversely

Proof: Follows from the fact that every closed set is $\psi^*\alpha$-closed.

Example 3.21 Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {a, b}, Y}. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity map. Then f is strongly $\psi^*\alpha$-closed but not quasi $\psi^*\alpha$-closed, since {b} is $\psi^*\alpha$-closed in (X, τ) but f({b}) = {b} is not closed in (Y, σ).

Proposition 3.22 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z,\eta)$ be any two maps.

(i) If f is quasi $\psi^*\alpha$-closed and g is $\psi^*\alpha$-closed then $g \circ f$ is strongly $\psi^*\alpha$-closed.

(ii) If f is strongly $\psi^*\alpha$-closed and g is quasi $\psi^*\alpha$-closed then $g \circ f$ is quasi $\psi^*\alpha$-closed.

(iii) If f and g are quasi $\psi^*\alpha$-closed maps then $g \circ f$ is strongly $\psi^*\alpha$- closed.

(iv) If f and g are strongly $\psi^*\alpha$-closed maps then $g \circ f$ is strongly $\psi^*\alpha$-closed.

Proof: (i) Let V be $\psi^*\alpha$ - closed in (X, τ). Since f is quasi $\psi^*\alpha$-closed, f(V) is closed in (Y, σ). Then $g(f(V)) = (g \circ f)(V)$ is $\psi^*\alpha$ - closed in (Z, η) as g is $\psi^*\alpha$ - closed. Hence $g \circ f$ is strongly $\psi^*\alpha$- closed.

(ii) Let V be $\psi^*\alpha$-closed in (X, τ). Since f is strongly $\psi^*\alpha$-closed, f(V) is $\psi^*\alpha$-closed in (Y, σ). Then $g(f(V)) = (g \circ f)(V)$ is closed in (Z, η) as g is quasi $\psi^*\alpha$ - closed. Hence $g \circ f$ is quasi $\psi^*\alpha$ - closed.

(iii) Since every closed set is $\psi^*\alpha$-closed, the result follows.

(iv) Let V be $\psi^*\alpha$-closed in (X, τ). Then f(V) is $\psi^*\alpha$-closed in (Y, σ), as f is strongly $\psi^*\alpha$-closed. Since g is strongly $\psi^*\alpha$-closed, $(g \circ f)(V) = g(f(V))$ is $\psi^*\alpha$-closed in (Z, η). Hence $g \circ f$ is strongly $\psi^*\alpha$-closed.

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