Dirac Equation in Complex Plane

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Abstract

Dirac equation was written in form of real derivatives for first order for electron and positron, from paper Energy equation in complex plane^[4] it was clear that energy of any matter or antimatter is complex, so there was a need of getting differential equation in complex derivatives. This paper gets a generic differential equation of electron and positron in complex differentials of single order.

Keywords–*Dirac* equation, Complex Number, Energy differential equation.

I. INTRODUCTION

Dirac equation is based on differential equations which are real derivatives as shown below ^[1].

$$(\beta m c^{2} + c(\alpha_{1} p_{1} + \alpha_{2} p_{2} + \alpha_{3} p_{3}))\psi(x,t)$$
$$= i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

But as per paper certainty principle in complex plane position and time is complex entity ^[2]. Hence there is a need of writing a differential equation in complex plane, which uses complex form of position and time in first order of differential equation.

II. Ψ IN COMPLEX FORM

.In Quantum mechanics ψ is written as shown below ^{[1].}

$$\Psi = e^{i(kx - \omega t)}$$

As we know electron and positron move in complex plane hence we need to modify the wave function such that both the planes are taken into consideration. Below wave function take both the pane into consideration.

$$\Psi = \Psi_o e^{i(kx_r - wt_r)} e^{i(kx_i - wt_i)} \operatorname{Eq1}$$

III. ENERGY OPERATOR IN COMPLEX PLANE

Currently energy operator is given by [3]

$$E = i\hbar \frac{\partial}{\partial t}$$

Above equation deals only in real plane, to get energy operator in complex plane we will have to take energy and time in complex plane.

We know from energy equation in complex plane^[4] energy of electron is given by.

$$E = mc^2 + i\hbar\omega$$

And energy of positron is given by

$$E = \hbar \omega + im c^2$$

To generalize above 2 equations we can take $mc^2 = \hbar \omega$

Then generalized energy equation would become

$$E = \hbar \omega + i\hbar \omega$$

If we take time in complex plane as $\tau = t_r + it_i$ and differentiate equation 1 w.r.t τ .

We get

$$\begin{split} \Psi &= \Psi_{o} e^{i(kx_{r} - wt_{r})} e^{i(kx_{i} - wt_{i})} \\ \frac{\partial \Psi}{\partial \tau} &= \frac{1}{2} \left(\frac{\partial}{\partial t_{r}} - i \frac{\partial}{\partial t_{i}} \right) \Psi_{o} e^{i(kx_{r} - \omega t_{r})} e^{i(kx_{i} - \omega t_{i})} \\ \frac{\partial \Psi}{\partial \tau} &= \frac{1}{2} \Psi_{o} e^{i(kx_{r} - wt_{r})} e^{i(kx_{i} - wt_{i})} \left(-i\omega - i(-i\omega) \right) \\ \frac{\partial \Psi}{\partial \tau} &= \frac{1}{2} \Psi \left(-i\omega - \omega \right) \right) \\ \frac{\partial \Psi}{\partial \tau} &= -\frac{1}{2} \Psi \left(\omega + i\omega \right) \\ \hbar \frac{\partial \Psi}{\partial \tau} &= -\frac{1}{2} \Psi \left(\hbar \omega + i\hbar \omega \right) \end{split}$$

multiply both side by \hbar

$$-2\hbar \frac{\partial \Psi}{\partial \tau} = (\hbar\omega + i\hbar\omega) \Psi$$

we know that $E = h\omega + ih\omega$ so

$$E \Psi = -2 \hbar \frac{\partial \Psi}{\partial \tau}$$

Thus energy operator equation in complex plane would be

$$E = -2\hbar \frac{\partial}{\partial \tau} \text{Eq } 2$$

IV. MOMENTUM OPERATOR IN COMPLEX PLANE

Momentum operator is given by ^[5]

 $p\,=\,-\,i\hbar\nabla$

As electron moves in real plane momentum operator can be written as

 $p = -i\hbar \nabla_r \text{ Eq } 3$

Where ∇_r is gradient in real plane.

And for positron momentum operator can be written as

 $p = -i\hbar \nabla_i \text{ Eq } 4$

V. DIRAC EQUATION IN COMPLEX PLANE

If we take magnitude of energy equation of positron and electron in complex plane we would get

$$\left|E\right| = \sqrt{\left(m\,c^{\,2}\right)^{\,2} + \left(\hbar\,\omega\right)^{\,2}}$$

Taking square on both sides we get

$$\left|E\right|^{2} = (m c^{2}) + (\hbar \omega)^{2}$$

We know that $mc^2 = \sqrt{m_o c^2 + (pc)^2}$

Substituting this in above equation of energy we get

$$\left|E\right|^{2} = (m_{o}c^{2}) + (pc)^{2} + (\hbar\omega)^{2}$$
We can spit above equation like Dirac did
$$\lambda = \begin{vmatrix}0 & 0 & -i\\0 & 0 & 0\end{vmatrix}$$

$$\begin{aligned} \left| E \right| \cdot \left| E \right| &= (c \left(\alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \alpha_3 \cdot p_3 \right) + \beta m_o c^2 + \lambda \hbar \omega) \\ (c \left(\alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \alpha_3 \cdot p_3 \right) + \beta m_o c^2 + \lambda \hbar \omega) \end{aligned}$$

$$\left| E \right| = (c(\alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \alpha_3 \cdot p_3) + \beta m_o c^2 + \lambda \hbar \omega)$$

Eq 5

For cross term to vanish following equations should satisfy.

$$\alpha^{2} = \beta^{2} = \lambda^{2} = I$$

$$\alpha_{i}\alpha_{j} + \alpha_{j}\alpha_{i} = 0 \quad \{i \neq j\}$$

$$\alpha_{i}\beta + \beta\alpha_{i} = 0$$

$$\alpha_{i}\lambda + \lambda\alpha_{i} = 0$$

$$\beta\lambda + \lambda\beta = 0$$

If we solve above equation we get following value of α_1 , α_2 , α_3 , β and λ

| α ₁ = | 0 | 0 | 0 | 1 |
|--------------------------|-----|----|----|-----|
| | 0 | 0 | 1 | 0 |
| | 0 | 1 | 0 | 0 |
| | 1 | 0 | 0 | 0 |
| | I | | | I |
| $\alpha_2 =$ | 0 | 0 | 0 | -i |
| | 0 | 0 | i | 0 |
| | 0 | -i | 0 | 0 |
| | i | 0 | 0 | 0 |
| | | | | |
| $\alpha_3 =$ | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | - 1 |
| | 1 | 0 | 0 | 0 |
| | 0 | -1 | 0 | 0 |
| | | | | |
| 1 | 0 | 0 | -i | 0 |
| $\lambda = \int_{1}^{1}$ | 0 | 0 | 0 | -i |
| | i 0 | | 0 | 0 |
| | 0 | i | 0 | 0 |
| | | | | |
| | 0 | 0 | -i | 0 |
| 2 _ | 0 | 0 | 0 | -i |
| λ = | i 0 | | 0 | 0 |
| | 0 | i | 0 | 0 |

We know from paper, energy equation in complex ^[4] plane real and imaginary part of energy is same, hence we can write energy in terms of magnitude as

$$E = \left| E \right| e^{i\frac{\pi}{4}}$$

If we multiply equation 5 with $e^{i\frac{\pi}{4}}$ we get

we know that
$$\hbar \omega \psi = i\hbar \frac{\partial \Psi}{\partial t}$$

$$E = (c(\alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \alpha_3 \cdot p_3) + \beta m_o c^2 + \lambda \hbar \omega) \cdot e^{i\frac{\pi}{4}} - 2\hbar \frac{\partial \psi}{\partial \tau} = ((c(\alpha_1 \cdot p_1 + \alpha_2 \cdot p_2 + \alpha_3 \cdot p_3) + \beta m_o c^2 + \lambda \hbar \omega) \cdot e^{i\frac{\pi}{4}} + \beta m_o c^2 \psi$$

Eq 6

 $+ \alpha_{2} \cdot p_{2} + \alpha_{3} \cdot p_{3})\psi$ $\psi + \lambda i\hbar \frac{\partial \Psi}{\partial t_i} . e^{i\frac{\pi}{4}}$) Eq 8

From Eq 3 we know that

$$p = -i\hbar\nabla$$

VI. DIFFERENTIAL EQUATION OF **ELECTRON**

If we replace Eq 2, Eq 3 and Eq 7 in Eq 6 we get

 $\hbar \omega$ Term in equation 6 can be written as

Because $E = \left| E \right| e^{i\frac{\pi}{4}}$

$$\frac{\partial \Psi}{\partial t_i} = \Psi_o e^{i(kx_r - \omega t_r)} e^{i(kx_i - \omega t_i)} (-\omega i)$$

$$\frac{\partial \Psi}{\partial t_i} = \Psi (-\omega i)$$

$$i\hbar \frac{\partial \Psi}{\partial t_i} = (\hbar \omega) \Psi$$

$$(\hbar \omega) \Psi = i\hbar \frac{\partial \Psi}{\partial t_i}$$
Eq 7

This is
$$p_i = -i\hbar \frac{\partial}{\partial x_i} \text{Eq } 9$$

If we replace Eq9 in Eq8 we get

$$-2\hbar \frac{\partial \psi}{\partial \tau} =$$

$$((c(\alpha_1.(-i\hbar \frac{\partial \psi}{\partial x_1}) + \alpha_2.(-i\hbar \frac{\partial \psi}{\partial x_2}) + \alpha_3.(-i\hbar \frac{\partial \psi}{\partial x_3}))$$

$$+\beta m_o c^2 \psi + \lambda i\hbar \frac{\partial \Psi}{\partial t_i}).e^{i\frac{\pi}{4}})$$

Divide by \hbar on both sides we get

$$-2 \frac{\partial \psi}{\partial \tau} =$$

$$((c (\alpha_1 . (-i \frac{\partial \psi}{\partial x_1}) + \alpha_2 . (-i \frac{\partial \psi}{\partial x_2})) + \alpha_3 . (-i \frac{\partial \psi}{\partial x_3}))$$

$$+ \beta \frac{m_o c^2}{\hbar} \psi + \lambda i \frac{\partial \Psi}{\partial t_i} . e^{i \frac{\pi}{4}})$$

$$\frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \left(\left(\left(c \left(\alpha_{1} \cdot \left(-i \frac{\partial \psi}{\partial x_{r1}} \right) + \alpha_{2} \cdot \left(-i \frac{\partial \psi}{\partial x_{r2}} \right) \right) + \alpha_{3} \cdot \left(-i \frac{\partial \psi}{\partial x_{r3}} \right) \right) + \beta \frac{m_{o} c^{2}}{\hbar} \psi + \lambda i \frac{\partial \Psi}{\partial t_{i}} \right) e^{i \frac{\pi}{4}} \right)$$

$$+\beta m_{o}c^{2}\psi +\lambda\hbar\omega\psi).e^{i\frac{\pi}{4}})$$

 $((c(\alpha_1.p_1 + \alpha_2.p_2 + \alpha_3.p_3)\psi))$

 $((c(\alpha_1.p_1 + \alpha_2.p_2 + \alpha_3.p_3) +$

 $\beta m_{o}c^{2} + \lambda \hbar \omega).e^{i\frac{\pi}{4}}\psi$

 $-2\hbar \frac{\partial \psi}{\partial \tau} =$

 $-2\hbar \frac{\partial \psi}{\partial \tau} =$

From above equation we can say that we are getting 4 types of electron but as per Dirac equation we got 2 types of electrons, which were spin up and spin down, the first two equation is similar to Dirac equations for electron but lower half has to be some sort of electron, being negative energy it can be termed as virtual electron spin up and spin down.

VII. DIFFERENTIAL EQUATION OF POSITRON

If we follow similar approach for positron we will get below equation

| $\frac{\partial \psi}{\partial \tau} = -\frac{1}{2} \left(\left(c \left(\alpha_1 \cdot \left(-i \frac{\partial \psi}{\partial x_{i1}} \right) + \alpha_2 \cdot \left(-i \frac{\partial \psi}{\partial x_{i2}} \right) \right) \right) \right)$ |
|---|
| $+\alpha_3.(-i\frac{\partial\psi}{\partial x_{i3}}))$ |
| $+\beta \frac{m_o c^2}{\hbar} \psi + \lambda i \frac{\partial \Psi}{\partial t_r}).e^{i\frac{\pi}{4}}))$ |

Here also we get 4 equations for positron; the lower 2 can be associated with positron and above two can be associated with virtual positron.

VIII. CONCLUSION

If we consider Dirac equation in complex plane we get electron, virtual electron, positron and virtual positron differential equation.

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