Advanced Generalized Fractional Kinetic Equation

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Abstract

The aim of present paper to obtain the solution of Advanced generalized fractional order kinetic equation involving the Miller-Ross function. The results obtained here is moderately universal in nature. Special cases, relating to the Mittag-Leffler function is also considered.

Keywords : *Fractional kinetic equation, Mittag-Leffler function, Riemann-Liouville operator, Laplace transform Miller-Ross function.*

Mathematics Subject Classification: *33C60, 33E12, 82C31, 26A33.*

I. INTRODUCTION

The Sun which is a big star is assumed to be in thermal equilibrium and hydrostatic equilibrium. To describe a model, we consider it is a spherical symmetric, self-gravitating non-rotating. The features of its are mass, luminosity, diameter, effective surface temperature, central temperature and density. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star like sun (Perdang [17] and Clayton [2]). Energy in such stars being produced by the process of chemical reactions.

Let us define an arbitrary reaction which is dependent on time $N = N(t)$. It is possible to calculate rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, then

$$\frac{dN}{dt} = -d + p.$$

The production or destruction at time t depends not only on N(t) but also on the previous history $N(t_1),\ t_1 < t$, of the variable N.

This was represented by (Haubold and Mathai [7])

$$dN/dt = -d(N_t) + p(N_t), \qquad (1)$$

Where $N(t)$ denotes the function defined by $N_t(t*) = N(t - t*), t* > 0$.

Haubold and Mathai [7] considered a special case of this equation, when spatial fluctuations or in homogeneities in quantity $N(t)$ are neglected, is given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t) \qquad (2)$$

Where the initial conditions are $N_i(t = 0) = N_0$ is the number density of species i at time $t = 0$; constant $c_i > 0$, is called standard kinetic equation.
The solution of the equation (2) is as follows:

$$N_i(t) = N_0\, e^{-c_i t} \qquad (3)$$

Or,

$$N(t) - N_0 = c\,{}_0^0D_t^{-1} N(t) \qquad (4)$$

As ${}_0^0D_t^{-1}$ is the integral operator, Haubold and Mathai [7] described the fractional generalization of the standard kinetic equation (2) as

$$N(t) - N_0 = c^{v}\,{}_0^0D_t^{-v} N(t) \qquad (5)$$

Where ${}_0^0D_t^{-v}$ is the Riemann-Liouville fractional integral operator; Miller and Ross [13]) defined by

$${}_0^0D_t^{-v} N(t) = \frac{1}{\Gamma(v)}\int_0^t (t - u)^{v-1} f(u) \quad R(v) > 0, \qquad (6)$$

The solution of the fractional kinetic equation (5) is given by (see Haubold and Mathai [7])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^{vk}}{\Gamma(vk+1)} (ct)^{vk} \qquad (7)$$

Also, Saxena, Mathai and Haubold [20] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which is the extension of the work of Haubold and Mathai [7].

In the present work we studied of the advanced generalized fractional order kinetic equation. The Advanced generalized fractional kinetic equation and its solution, obtained in terms of the Miller-Ross function.

II. THE MILLER-ROSS FUNCTION

Miller and Ross [13] (1993, pp80 and 309-351) introduce this function which is defined as follows:

$$E_t(v,a) = \sum_{k=o}^{\infty} \frac{a^k t^{k+v}}{\Gamma(v+k+1)} \tag{8}$$

III. ADVANCED GENERALIZED FRACTIONAL KINETIC EQUATIONS

In this section, we investigate the solution of advanced generalized fractional order kinetic equation. The results are obtained in a compact form in terms of Miller-Ross function. The result is presented in the form of a theorem as follows:

Theorem 1:

If $v > 0, c > 0, b > 0, d > 0$ and $c \neq d$ then for the solution of the Advanced generalized fractional order kinetic equation

$$N(t) - N_0 E_{dt}(v,a) = -b^{-v} c^v {}_0^0D_t^{-v} N(t) \tag{9}$$

Then

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k b^{-vk} d^{k+v} (c)^{kv} \; E_t(vk+v,a) \tag{10}$$

Proof. Applying the Laplace transform both the sides of equation (9), we get

$$L\{N(t)\} - L\{N_0 E_{dt}(v,a)\} = L\{-b^{-v} c^v {}_0^0D_t^{-v} N(t)\}$$

$$N(s) - N_0 \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}} = -b^{-v} c^v s^{-v} N(s) \tag{11}$$

Or,

$$N(s) = \frac{N_0}{(1 + b^{-v} c^v s^{-v})} \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}} \tag{12}$$

Now, taking inverse Laplace transform both the sides of (12), we get

$$L^{-1}\{N(s)\} = L^{-1}\left\{N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (b^{-v} c^v s^{-v})^k (1)_k}{k!} \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}}\right\} \tag{13}$$

Or

$$N(t) = N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (b^{-v} c^v)^k (1)_k}{k!} \frac{d^{k+v} a^k t^{vk+v+k}}{\Gamma(vk+v+k+1)}$$

Or,

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k b^{-vk} d^{k+v} (c)^{kv} E_t(vk+v,a) \tag{14}$$

This is the complete proof of the theorem (9).

IV. SPECIAL CASES

When $a = 1, v = 1$ then

Corollary: 1. If $v > 0, c > 0$ then for the solution of the Advanced generalized fractional kinetic equation (in form of Mittag-Leffler function [10])

$$N(t) - N_0 \, t \, E_{1,2}(t) = -b^{-v} c^v {}_0^0D_t^{-v} N(t) \tag{15}$$

There holds the formula

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k b^{-vk} (c)^{kv} \; t \, E_{2,2}(t^2) \tag{16}$$

Corollary: 2. If $v > 0, c > 0, b = 1$ then for the solution of the Advanced generalized fractional kinetic equation again (in terms of Mittag-Leffler function [10])

$$N(t) - N_0 \, t \, E_{1,2}(t) = -c^v {}_0^0D_t^{-v} N(t) \tag{15}$$

There holds the formula

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k (c)^{kv} \; t \, E_{2,2}(t^2) \tag{16}$$

Corollary: 3. If $v > 0, c = 1, b = 1$ then for the solution of the Advanced generalized fractional kinetic equation (in form of Mittag-Leffler function [10])

$$N(t) - N_0 \, t \, E_{1,2}(t) = -{}_0^0D_t^{-v} N(t) \tag{15}$$

There holds the formula

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k \; t \, E_{2,2}(t^2) \tag{16}$$

Remark: If we put $c = 1, v = 2$ and $\mu = 2$ in paper of Saxena , Haubold amd Mathai(2002)[20] The result (16) is same as obtained by Saxena, Haubold amd Mathai (2002) in solving a theorem (1) for generalized fractional kinetic equation.

Theorem-2. If $\nu > 0, c > 0, b > 0, d > 0$ and $c = d$ then for the solution of the Advanced generalized fractional order kinetic equation

$$N(t) - N_0 E_{dt}(v,a) = -b^{-v} d^{v}\,{}_0^0D_t^{-v}\, N(t) \quad (17)$$

Then

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k b^{-vk}\; E_{dt}(vk+v,a) \quad (18)$$

Proof. We solve equation (17) as term by term previous theorem, applying the Laplace transform both the sides, we get

$$L\{N(t)\} - L\,\{N_0 E_{dt}(v,a)\} = L\{-b^{-v} d^{v}\,{}_0^0D_t^{-v}\, N(t)\}$$

$$N(s) - N_0 \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}} = -b^{-v} d^{v} s^{-v} N(s) \quad (19)$$

Or,

$$N(s) = \frac{N_0}{(1 + b^{-v} d^{v} s^{-v})} \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}} \quad (20)$$

Now, taking inverse Laplace transform both the sides of (20), we get

$$L^{-1}\{N(s)\}$$

$$= L^{-1}\left\{ N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (b^{-v} d^{v} s^{-v})^k (1)_k}{k!} \sum_{k=o}^{\infty} \frac{d^{k+v} a^k}{s^{k+v+1}} \right\} \quad (21)$$

Or

$$N(t) = N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (b^{-v})^k (1)_k}{k!} \frac{a^k (dt)^{vk+v+k}}{\Gamma(vk+v+k+1)}$$

Or,

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k b^{-vk}\; E_{dt}(vk+v,a) \quad (22)$$

Special case: If $\nu > 0,\ d > 0, b = 1$ then for the solution of the Advanced generalized fractional order kinetic equation

$$N(t) - N_0 E_{dt}(v,a) = -b^{-v} c^{v}\,{}_0^0D_t^{-v}\, N(t) \quad (23)$$

Then holds the result

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k d^{k+v} (c)^{kv}\; E_t(vk+v,a) \quad (24)$$

V. CONCLUSION

In this present work, we have introduced an advanced fractional generalization of the standard kinetic equation and established solution for the same. The Advanced fractional kinetic equation discussed in this paper, involving Miller-Ross function contains a number of known (may be new also) results. Advanced fractional kinetic equations are involving also Mittag-Leffler functions.

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