# The Brachistochrone Problem: Relative Study of Straight Line Path with Minimum Time Path

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***Abstract***
*The brachistochrone problem under influence of gravity has history of science and mathematics. The two points, in vertical plane, higher and lower are fixed then the solution of brachistochrone problem is a segment of a cycloid. I considered lower point is movable along cycloid. The time difference, along cycloid and corresponding straight line path, is compaired during first half cycle. In fact, the path of particle is constrained and hence part of energy is lost by the particle due to kinetic friction and air resistance.*

**Keywords**- *The brachistochrone, The cycloid, the generating circle.*

## I. HISTORY

One of the well-known the brachistochrone problem is to find shape of curve joining two points, along which a particle falling from rest under influence of gravity travels from the higher to lower points in the least time. The term derives from ***Greek brachistos*** – 'the ***shortest' and chronos – 'time, delay'***.

The brachistochrone problem was one of the earliest problems posed in the ***calculus of variations***. Newton was challenged to solve the problem in 1696, and did so the very next day [1]. The path of quickest descent from one point to another point is not shortest path, namely ***straight line***, but the ***arc of circle*** [2]. Erlichson [8] argues that Galileo restricted himself to descent paths that used points along a circle. According to Nahin [3], Galileo's claim is correct, but his reasoning was flawed. ***The cycloid*** was discovered in the early sixteenth century by the mathematician Charles Bouvelles [4]. In1659, Huygens discovered that the cycloid is a solution to the ***isochrone*** and will exhibit simple harmonic motion with period independent of the starting point [5]. Johann Bernoulli solved the problem using the analogous one of the path considering the path of light refracted by transparent layers of varying density [6], [7]. Actually, ***Johann curve is cycloid***, and challenged his brother Jakob to find required curve. When Jakob correctly did so, Johann tried to substitute the proof his own [1].

Most of these discoveries concerning the cycloid are inaccessible to students with no calculus background. But construction of cycloid using generating circle and straight line paths is quite accessible to students. The brachistochrone problem is taught under graduate and post graduate students in the subject-***classical mechanics***.

## II. THE BRACHISTOCHRONE PROBLEM

In the solution, the particle of mass m may actual travel under the influence of gravity along the cycloid for a distance, but the path is nonetheless faster than straight line.

The time taken by the particle from a point to another point is given by

$$t12 = \int_1^2 \frac{ds}{v}$$

Where ds is ***arc length*** and $v$ is ***speed***.

The speed at any point is given by principle of conservation of energy.

$$\frac{1}{2}mv^2 = mgy \qquad \text{therefore} \quad v=\sqrt{2gy}$$

and $$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \sqrt{1+y'^2}\,dx \qquad \text{then gives}$$

$$t12 = \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx = \int_1^2 f\,dx$$

$$\text{Where } f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} = f(y, y')$$

The time $t_{12}$ will be minimum, the Euler-Lagranges equation must be satisfied. The $f$ is not function of x, then for special case of $\frac{\partial f}{\partial x}$ =0, the Euler–Lagranges equation $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$ reduces to the Baltrami identity,

$$f - y'\frac{\partial f}{\partial y'} = c\text{, constant,}$$

Therefore, $$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy}\sqrt{1+y'^2}}$$

substract $y'\frac{\partial f}{\partial y'}$ from $f$ and simplifying, then gives,

$$\frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

squaring both sides and rearranging, we get,

$$(1 + y'^2)y = \frac{1}{2gc^2} = 2a$$

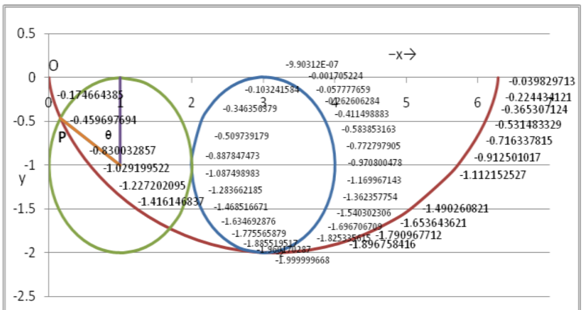
This equation is solved by parametric equations, the well known result is obtained

$$x = a(\theta - \sin\theta)$$
$$y = -a(1 - \cos\theta)$$

Where a=**radius of generating circle** of cycloid.

A cycloid is curve traced by a point on the circumference of a circle as the circle rolls, without slipping along a straight line.

For a=1m, cycloid is drawn in figure (1). Few values of y co-coordinates are also shown in same figure.




radius of generating circle a=1m

**Figure -1**

My approach is that the particle 'P' moves on cycloid from origin to extreme point.

## III. THE TIME TAKEN BY PARTICLE ALONG CYCLOID FROM HIGHER POINT (ORIGIN) TO LOWER POINT.

$$t1 = \int_o^P \frac{ds}{v} = \int_o^P \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}dx$$

Here consider magnitude of y only.

We have

$$x = a(\theta - \sin\theta)$$
$$y = -a(1 - \cos\theta)$$

$dx = a(1 - \cos\theta)d\theta$ , $dy = -a\sin\theta d\theta$

therefore $\frac{dy}{dx} = -\cot(\theta/2)$

The radius of generating circle is taken a=1 m. The point p is movable, so upper limit is kept $\theta$

$$t1 = \frac{1}{\sqrt{2g}} \int_o^\theta \frac{\sqrt{1 + \cot^2\left(\frac{\theta}{2}\right)}}{\sqrt{1 - \cos\theta}} (1 - \cos\theta)d\theta$$

$$= \frac{1}{\sqrt{2g}} \int_o^\theta \sqrt{2}\, d\theta = \frac{\theta}{\sqrt{g}} \text{ ------------- (1)}$$

$0 \le \theta \le 2\pi$ , $g = 9.8\, m/s2$

## IV. THE TIME TAKEN BY PARTICLE ALONG STRAIGHT LINE PATH.

$$t2 = \int_o^P \frac{ds}{v} = \int_o^P \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\sqrt{2gy}}dx$$

y=m1x , equation of straight line passes through origin.

Therefore $m1 = \left(\frac{dy}{dx}\right) = \frac{y}{x}$

let x be the co-ordinate corresponding to point P.

$$t2 = \frac{1}{\sqrt{2g}} \int_o^x \frac{\sqrt{1 + m1^2}}{\sqrt{m1x}} dx$$

but $m1 = \frac{dy}{dx}$=constant for straight line.

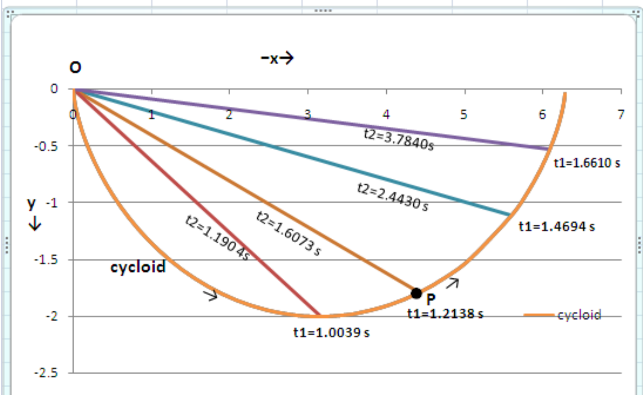
$$t2 = \frac{\sqrt{1 + m1^2}}{\sqrt{2g}\sqrt{m1}} \int_o^x \frac{dx}{\sqrt{x}}$$

after solving

$$t2 = \frac{2\sqrt{x^2 + y^2}}{\sqrt{2gy}}$$

$$= \frac{2s}{\sqrt{2gy}} \text{ ----------------- (2)}$$

From equations (1) and (2), the times taken by particle, along cycloid and corresponding straight line paths, were calculated from 0 to $2\pi$ with interval of 0.2 radian and for a=1m using Microsoft Excel. The particle moves along cycloid and the co-ordinates (x,y) are taken corresponding to angle $\theta$ made by generating circle. The few straight line paths are shown in figure (2).




radius of generatig circle a=1m.

**figure -2**

## V. THE TIME DIFFERENCE.

The graphs of $t1, t2$ against x co-ordinate are shown in figure (3).

From equations (1) & (2)

$$t2 - t1 = \frac{2s}{\sqrt{2gy}} - \frac{\theta}{\sqrt{g}} \text{ ----------------- (3)}$$

Difference, $(t2 - t1)$ is shown in figure (3).

As 'P' approaches to lowest point of cycloid [figure (2)], there is no much difference between times[figure (3)]. As 'P' moves towards extreme point during first half cycle, the time difference increases approximate exponentially and 'P' tends to extreme point,

$$(t2 - t1) \rightarrow \infty(infinity)$$

The particle does not take straight line (y=0) path. Along y=0, there is no exchange of energy.

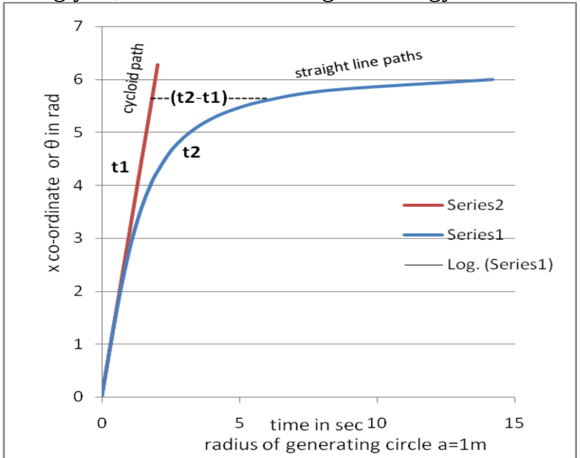



Figure-3

If the path of particle is restricted by wire or cycloid surface (any surface), we cannot avoid kinetic friction and air resistance. So I performed the experiment by taking cycloid of stainless steel surface and steel metallic ball of diameter 0.625 cm and mass 1.05 g. The maximum recoil vertical height gained by ball was 0.64 times original height. The energy lost by ball in first half cycle was 36%.

## VI. CONCLUSIONS:

The time difference taken by particle to fall under the influence of gravity, between cycloid path and corresponding straight line path increases exponential from initial starting point (higher) to extreme point during first half cycle. The particle does not take horizontal straight line path because line is equipotential & there is no exchange of energy along the line. The energy lost by ball along cycloid during first half cycle was 36% due to kinetic friction.

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