Inner Relationship among Rapidity, Velocity and Geometric Approach to the Wigner Rotation

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Abstract

Rapidity is a hyperbolic angle that differentiates two frames of reference in relative motion. We demonstrate how this space can be calculated to get various effects resulting from the successive application of non-collinear Lorentz boosts and the relativistic addition of non-collinear velocities. We are going to observe the relation between rapidity and velocity of a moving particle. It has been explained how rapidity space provides a geometric approach to the Wigner rotation and the Thomas precession. We have also explained that Thomas-Wigner rotation occurs due to boost angle θ and velocity.

Keywords - *Hyperbolic Angle, Rapidity, Lorentz Transformation, Wigner Rotation.*

I. INTRODUCTION

Let us consider two inertial frames of references S and S′ where the frame S is at rest and the frame S′ is moving along x-axis with velocity v with respect to S frame. When Lorentz transformation preserves the orientation of the spatial axes of S then it is called a boost. In some cases, an inertial frame S′ is obtained from an inertial frame S by two successive boosts. If the two successive boosts are non-collinear, then the single Lorentz transformation is the resultant which is not a pure boost. But it is the product of a boost and a rotation. The unexpected rotation discovered by Thomas [1] in 1926 and derived by Wigner [2] is called Wigner rotation. If successive non-collinear boosts return the spatial origin of S′ to the spatial origin of S, then all the Thomas-Wigner rotations along the way combine to produce a net rotation of S′ with respect to S called the Thomas precession [5-10].

A relativistic velocity space is called rapidity space and we have derived how rapidity space provides a geometric approach to the Wigner rotation and the Thomas precession. We have also observed the relation between rapidity and velocity of a moving particle. Moreover, we derive the equation of Wigner rotation in hyperbolic space.

II. LORENTZ TRANSFORMATION

The transformation which relates the observations of position and time made by the two observers in two different inertial frames is known as Lorentz transformation.

A. Special Lorentz Transformation

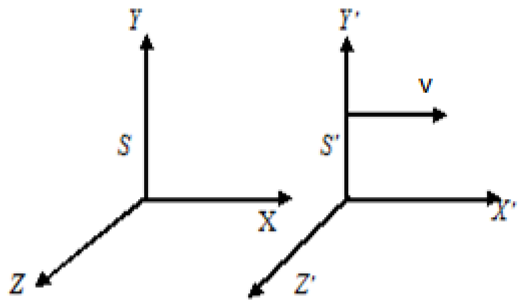

Figure 1: The frame S is at rest and the frame S′ is moving with respect to S with uniform velocity v along x-axis.

In the special theory of relativity, consider two inertial frames of references S and S′ where space and time coordinate (x, y, z, t) and (x′, y′, z′, t′) respectively. The relation between the coordinates of S and S′ is called special Lorentz transformation. It can be written as [1, 3-7]

$$\left.\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ \text{Where, } \gamma &= \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \end{aligned}\right| \quad (1)$$

B. Special Lorentz Transformation In Hyperbolic Space And Rapidity

Rapidity can be defined as the hyperbolic angle that differentiates two frames of reference in relative motion. Each frame is associated with distance and time coordinates. It is commonly used as a measure for relativistic velocity in hyperbolic space. The rapidity ϕ in terms of the velocity v can be written as,

$$v = \tanh\phi \quad (2)$$

So, $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-tanh^2}} = \cosh\phi \quad (3)$

and $\gamma v = \cosh\phi \times \tanh\phi$

$$\gamma v = \sinh\phi \quad (4)$$

Lorentz transformations can be expressed in a way that looks similar to circular motion in hyperbolic space using the hyperbolic function. Equation (1) can be written as,

$$x' = cosh\phi x - sinh\phi t$$
$$y' = y \quad (5)$$
$$z' = z$$
$$t' = cosh\phi t - sinh\phi x$$

If we rotate our coordinates between x, t, and x', t' we will obtain a diagram as in Figure 2.

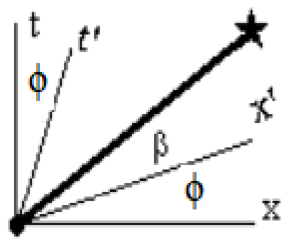

Figure 2: Geometrical coordinate rotation between x, t, and x', t' for hyperbolic trigonometry.

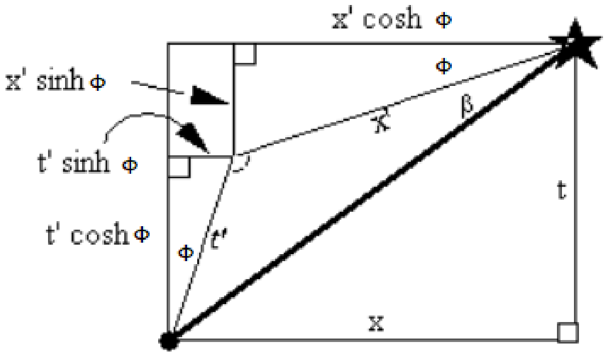

Figure 3: Geometrical relationship between x, t, and x', t' for hyperbolic trigonometry.

From figure 3, it is seen that

$$x = x' cosh\phi + t' sinh\phi \quad (6)$$
$$t = x' sinh\phi + t' cosh\phi$$

Reversing the equations (6) gives

$$x' = xcosh\phi - tsinh\phi$$

(7)

$$t' = -xsinh\phi + tcosh\phi$$

In hyperbolic trigonometry equation (7) it can be written as,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} cosh\phi & -sinh\phi \\ -sinh\phi & cosh\phi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \quad (8)$$

Hyperbolic angle ϕ is called rapidity which is determined by using equation (2).

$$\phi = \tanh^{-1} v$$

Let us consider the particle has different velocities and corresponding rapidity values calculated in the following table.

Velocity in unit of c	Rapidity
0.1	0.10
0.2	0.20
0.3	0.31
0.4	0.42
0.5	0.55
0.6	0.69
0.7	0.87
0.8	1.1
0.9	1.5
0.925	1.6
0.950	1.8
0.975	2.2
0.99	2.7

Table 1: Different velocities and corresponding rapidity values

Using the data from the table a graph of rapidity versus velocity of moving particle is plotted which is shown in figure 4.

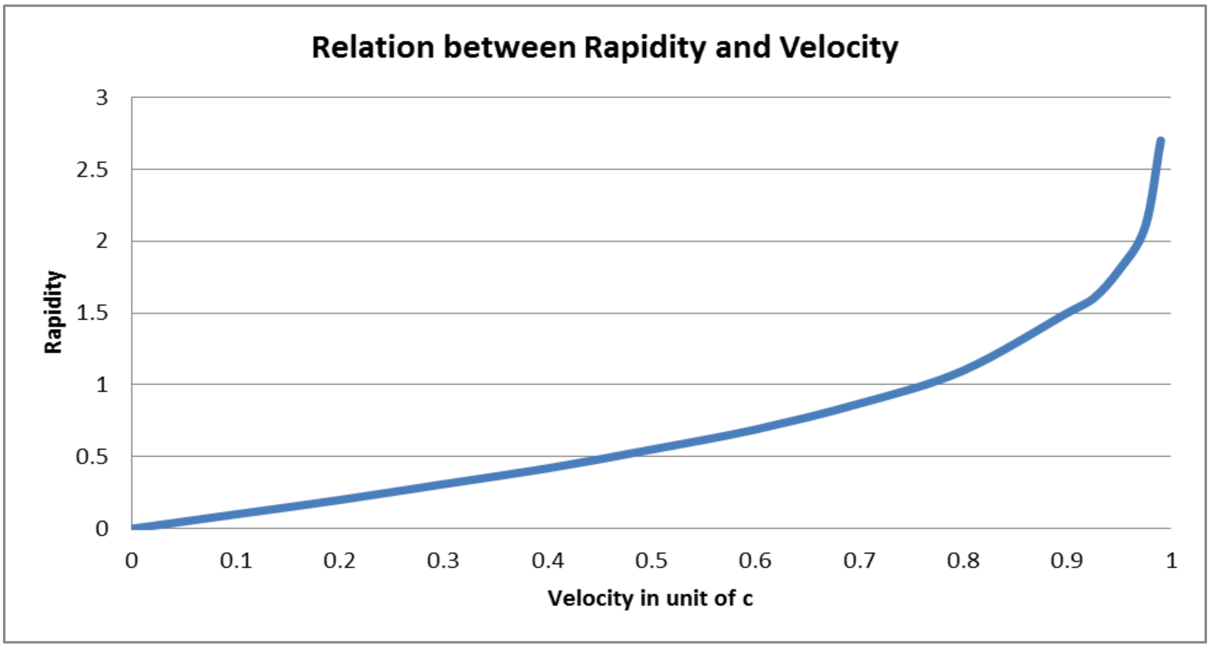

Figure 4: Relation between Rapidity and Velocity.

In the graph of the rapidity versus velocity of the moving particle we have experienced that the rapidity slowly increases due to the increase of the velocity of the particle up to 0.8c. When velocity is 0.8c to 0.99c, rapidity increases sharply.

III. GEOMETRIC APPROACH TO THE WIGNER ROTATION

A boost from one velocity to another is represented in rapidity space by the geodesic connecting them because a pure boost does not involve any rotation of the reference frame. A boost with rapidity Φ_1 in the *x* direction is followed by a boost with rapidity ϕ_2 in the $\theta = \pi - \alpha_1$ direction. Being boosted the coordinate axes representing the boost in rapidity space maintain a fixed angle $\pi - \alpha_1$ with respect to the geodesic that they follow, as shown in figure 5. Let us consider geodesic lies on the arc of a circle. As shown in figure 5, cross hairs moving along such type of geodesic maintain their orientation with respect to it [7-8].

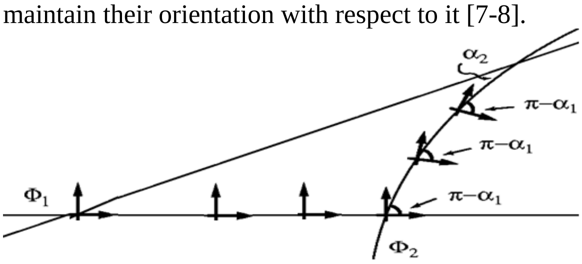

Figure 5: The first boost along the geodesic Φ_1 and second one along the geodesic Φ_2

Here, the cross hairs move back to the origin along a closed path that includes one of these geodesics which will be rotated with respect to their initial orientation. An example of this rotation is shown in figure 6.

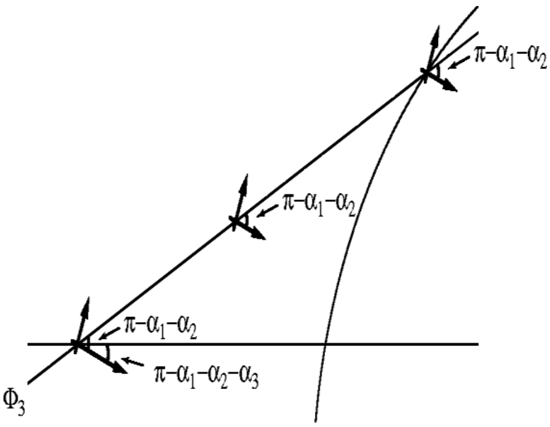

Figure 6: The third boost along the geodesic Φ_3

In figure 6, the straight-line geodesic making an angle α_3 with the horizontal axis is called Φ_3, and the other two geodesics are Φ_1 and Φ_2. We see in figure 5, when we boost from the origin along Φ_1, there is no change in the orientation of the crosshairs. When we boost along Φ_2, the x-axis of the crosshairs maintains its angle $\pi - \alpha_1$ with respect to Φ_2, because Φ_1 is a geodesic. Finally, as shown in figure 6, boosting back to the origin along Φ_3, the x-axis of the crosshairs maintains its orientation of $(\pi-\alpha_1)-\alpha_2$ with respect to the geodesic Φ_1. When the coordinate system returns to the rapidity space origin, its x-axis will have rotated from its initial orientation in the clockwise direction by the Thomas-Wigner rotation angle.

$$TWR = -[\pi - (\alpha_1 - \alpha_2 - \alpha_3)]$$

$$\omega = -[\pi - (\alpha_1 - \alpha_2 - \alpha_3)] \qquad (9)$$

The absolute value of the right-hand side of equation (9) is known as Wigner rotation. We assume that the area enclosed by the rapidity space triangle is proportional to the TWR angle.

IV. DERIVATION OF THE EQUATION OF THE WIGNER ROTATION IN HYPERBOLIC SPACE

Lorentz transformations work in hyperbolic space using the hyperbolic function of equation (6), a boost in the positive x direction with rapidity**ϕ.** The action of the boost can be expressed in a particularly simple way as a Mobius transformation.

$$z \to z' = \frac{az + b}{bz + a}$$

Another special type of Lorentz transformation which is easy to analyse is a spatial rotation. It is not difficult to see that a counter clockwise spatial rotation by an angle produces the map of the disk

$$z \to z' = e^{i\theta} z = \frac{e^{i\theta/2 + 0}}{0z + e^{i\theta/2}} \qquad (12)$$

Using Mobius transformation equation of equation (12) the rotation metric in rapidity space can be written as,

$$R(\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \qquad . \ (13)$$

The boost of equation (6) in rapidity space which can be written as,

$$B_x(\phi) = \begin{pmatrix} \cosh\frac{\phi}{2} & -\sinh\frac{\phi}{2} \\ -\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix} \qquad .... \ (14)$$

A pure boost with rapidity ϕ in the direction of θ can be obtained by first rotating through −θ. Then it applies an x-boost of ϕ. Then it rotates back by θ. To show these three operations with matrices, we have

$$R(\theta)B_x(\phi)R(-\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\begin{pmatrix} \cosh\frac{\phi}{2} & -\sinh\frac{\phi}{2} \\ -\sinh\frac{\phi}{2} & \cosh\frac{\phi}{2} \end{pmatrix}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$R(\theta)B_x(\phi)R(-\theta) = \begin{pmatrix} \cosh\frac{\phi}{2}\times e^{i\theta/2} + 0\times -\sinh\frac{\phi}{2} & e^{i\theta/2}\times\left(-\sinh\frac{\phi}{2}\right) + 0\times\cosh\frac{\phi}{2} \\ 0\times\cosh\frac{\phi}{2} + e^{-i\theta/2}\times -\sinh\frac{\phi}{2} & 0\times\left(-\sinh\frac{\phi}{2}\right) + \cosh\frac{\phi}{2}\times e^{-i\theta/2} \end{pmatrix}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$R(\theta)B_x(\phi)R(-\theta) = \begin{pmatrix} e^{i\theta/2}\cosh\frac{\phi}{2} & e^{i\theta/2}\left(-\sinh\frac{\phi}{2}\right) \\ e^{-i\theta/2}\left(-\sinh\frac{\phi}{2}\right) & \cosh\frac{\phi}{2}\, e^{i\theta/2} \end{pmatrix}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$R(\theta)B_x(\phi)R(-\theta) = \begin{pmatrix} e^{i\theta/2}\cosh\frac{\phi}{2}\times e^{-i\theta/2} - \sinh\frac{\phi}{2} e^{i\theta/2}\times 0 & e^{i\theta/2}\cosh\frac{\phi}{2}\times 0 - e^{i\theta/2}\times\sinh\frac{\phi}{2} e^{i\theta/2} \\ e^{-i\theta/2} - \sinh\frac{\phi}{2}\times e^{-i\theta/2} + 0\times e^{i\theta/2}\cosh\frac{\phi}{2} & -\sinh\frac{\phi}{2} e^{-i\theta/2}\times 0 + e^{i\theta/2}\cosh\frac{\phi}{2} e^{i\theta/2} \end{pmatrix}$$

$$R(\theta)B_x(\phi)R(-\theta) = \begin{pmatrix} \cosh\frac{\phi}{2} & -\sinh\frac{\phi}{2} e^{i\theta} \\ -\sinh\frac{\phi}{2} e^{-i\theta} & \cosh\frac{\phi}{2} \end{pmatrix} \qquad \ldots\ldots\ldots\ldots \ (15)$$

Therefore, as shown in figure 6, a boost with a rapidity of ϕ_1 in the x direction followed by a boost of rapidity ϕ_2 in the $\theta = \pi - \alpha_1$ direction, corresponds to

$$R(\theta)B_x(\phi_2)R(-\theta)B_x(\phi_1) = \begin{pmatrix} \cosh\frac{\phi_2}{2} & -\sinh\frac{\phi_2}{2} e^{i\theta} \\ -\sinh\frac{\phi_2}{2} e^{-i\theta} & \cosh\frac{\phi_2}{2} \end{pmatrix}\begin{pmatrix} \cosh\frac{\phi_1}{2} & -\sinh\frac{\phi_1}{2} \\ -\sinh\frac{\phi_1}{2} & \cosh\frac{\phi_1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\frac{\phi_1}{2}\cosh\frac{\phi_2}{2} + \sinh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2}e^{i\theta} & -\sinh\frac{\phi_1}{2}\cosh\frac{\phi_2}{2} - \cosh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2}e^{i\theta} \\ -\cosh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2}e^{-i\theta} - \sinh\frac{\phi_1}{2}\cosh\frac{\phi_2}{2} & \sinh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2}e^{-i\theta} + \cosh\frac{\phi_1}{2}\cosh\frac{\phi_2}{2} \end{pmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \ (16)$$

On the other hand, any Lorentz transformation in the direction of ω_1 can be expressed as the product of a boost $R(\omega_1)B_x(\phi_3)R(-\omega_1)$ in the ω_1 direction followed by a rotation through an angle ω_2 . To express these operations with matrices we have

$$R(\omega_2)(R(w_1)B_x(\phi_3)R(-w_1)$$

$$= \begin{pmatrix} e^{\frac{i\omega_2}{2}} & 0 \\ 0 & e^{\frac{-i\omega_2}{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{i\omega_1}{2}} & 0 \\ 0 & e^{\frac{-i\omega_1}{2}} \end{pmatrix}\begin{pmatrix} \cosh\frac{\phi_3}{2} & -\sinh\frac{\phi_3}{2} \\ -\sinh\frac{\phi_3}{2} & \cosh\frac{\phi_3}{2} \end{pmatrix}\begin{pmatrix} e^{\frac{-i\omega_1}{2}} & 0 \\ 0 & e^{\frac{i\omega_1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{i\omega_2}{2}}\times e^{\frac{i\omega_1}{2}} & 0 \\ 0 & e^{\frac{-i\omega_2}{2}}\times e^{\frac{-i\omega_1}{2}} \end{pmatrix}\begin{pmatrix} \cosh\frac{\phi_3}{2}\times e^{\frac{-i\omega_1}{2}} & -\sinh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \\ -\sinh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} & \cosh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{i(\omega_1+\omega_2)}{2}} & 0 \\ 0 & e^{\frac{-i(\omega_1+\omega_2)}{2}} \end{pmatrix}\begin{pmatrix} \cosh\frac{\phi_3}{2}\times e^{\frac{-i\omega_1}{2}} & -\sinh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \\ -\sinh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} & \cosh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{i(\omega_1+\omega_2)}{2}}\times\cosh\frac{\phi_3}{2}\times e^{\frac{-i\omega_1}{2}} & e^{\frac{i(\omega_1+\omega_2)}{2}}\times -\sinh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \\ -\sinh\frac{\phi_3}{2}\times e^{\frac{-i\omega_1}{2}}\times e^{\frac{-i(\omega_1+\omega_2)}{2}} & e^{\frac{-i(\omega_1+\omega_2)}{2}}\times\cosh\frac{\phi_3}{2}\times e^{\frac{i\omega_1}{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\frac{\phi_3}{2}\times e^{\frac{i\omega_2}{2}} & -\sinh\frac{\phi_3}{2}\times e^{i\omega_1}\times e^{\frac{i\omega_2}{2}} \\ -\sinh\frac{\phi_3}{2}\times e^{-i\omega_1}\times e^{\frac{-i\omega_2}{2}} & \cosh\frac{\phi_3}{2}\times e^{\frac{-i\omega_2}{2}} \end{pmatrix}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \ (17)$$

The upper left part of equation (16) and (17) can be written as,

$$\omega_2 = 2\arg\left(1 + \tanh\frac{\phi_1}{2}\tanh\frac{\phi_2}{2}e^{i\theta}\right) \qquad .... \ (18)$$

Equation (18) is an algebraic formula for the Thomas– Wigner rotation ω_2 resulting from a boost with rapidity ϕ_1 in the *x* direction followed by a boost with rapidity ϕ_2 in the $\theta = \pi - \alpha_1$ direction.

V. NUMERICAL VALUES OF WIGNER ROTATION IN HYPERBOLIC SPACE

In equation (18) we have used different boost velocities and boost angles to measure Wigner rotation. The findings are as in the table below. Using the data from the table a graph of Wigner rotation vs. boost angle is plotted nest to table.

Boost Velocity v_1 and v_2 in unit of c	Rapidity ϕ_1 & ϕ_2	Boost Angle θ (deg)	Wigner Rotation ω_2 (deg)
$v_1 = 0.4$c and v_2=0.5c	ϕ_1=0.4236 and $\phi_2 = 0.5493$	0	0
		30	3.05
		60	5.39
		90	6.40
		120	5.70
		150	3.37
		180	0
$v_1 = 0.6$c and v_2=0.7c	ϕ_1=0.6931 and $\phi_2 = 0.8673$	0	0
		30	6.97
		60	12.60
		90	15.50
		120	14.41
		150	8.82
		180	0
$v_1 = 0.8$c and v_2=0.9c	ϕ_1=1.099 and $\phi_2 = 1.472$	0	0
		30	14.05
		60	26.41
		90	34.80
		120	35.68
		150	24.28
		180	0

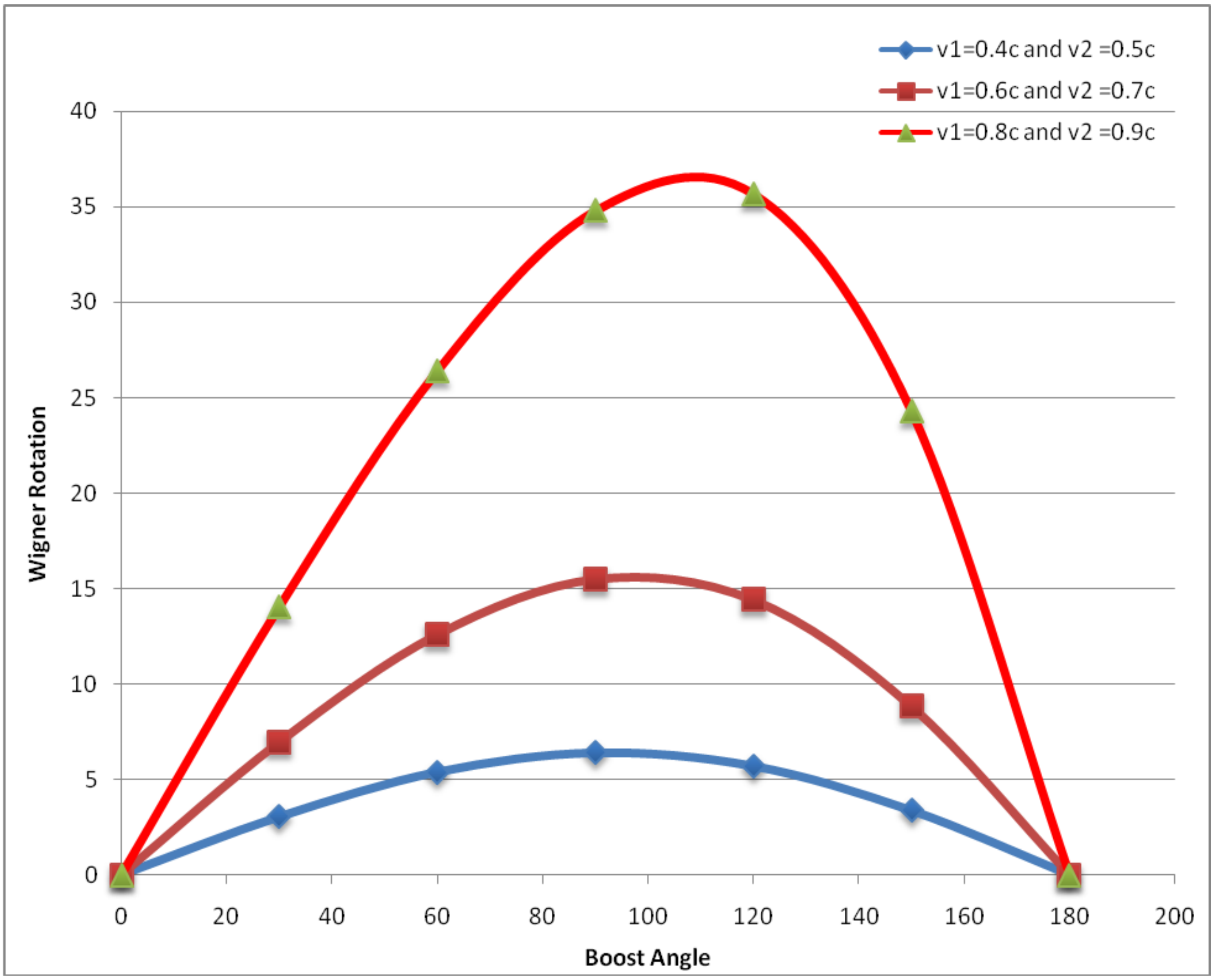

Figure 7: Wigner rotation changes due to different boost velocities and boost angles.

VI. CONCLUTION

Wigner rotation and rapidity have been presented. We have also explained the relation between rapidity and velocity of a moving particle. It has been observed that the rapidity increases due to the increase of velocity. We have demonstrated how rapidity space provides a geometric approach to the Wigner rotation and the Thomas precession. Moreover, we have derived the equation of Wigner rotation in hyperbolic space. Thomas-Wigner rotation occurs due to boost angle θ and boost velocities which have been clearly depicted. In figure 8, it is observed that the Wigner rotation increases due to the increase of boost velocities v_1, v_2. At boost velocity v_1=0.8c and v_2=0.9c, Wigner rotation increases very fast up to boost angle 110^o and then it falls down sharply. When the two boost velocities are v_1=0.6c and v_2=0.7c, Wigner rotation increases slowly up to boost angle 100^o and then it falls down gradually. When the boost velocities are v_1=0.4c and v_2=0.5c, Wigner rotation increases very slowly up to boost angle 90^o and then it decreases very slowly.

REFERENCES

[1] L.H.Thomas, “Motion of the spinning electron,” Nature 117, 514, 1926; ibid., “The Kinematics of an electron with an axis,” Phil. Mag. 3, 1–23, 1927.
[2] E.P.Wigner, “On unitary representations of the inhomogeneous Lorentz group,” Ann. Math. 40, 149–204 (1939).
[3] G.Beyerle “Visualization of Thomas–Wigner Rotations”, Symmetry, 9, 292, (2017).
[4] J.P.Lambare “Fermi–Walker transport and Thomas precession”, Eur. J. Phys. 38, 045602, (2017).
[5] M.T.Hossain, M. S. Alam “Numerical Calculations of the Wigner rotation” International Journal of Physics and Research 7(2):1-16, (2017).
[6] M.T.Hossain, M. S. Alam “Transformation of Orbital Angular Momentum and Spin Angular Momentum” American Journal of Mathematics and Statistics 6(5): 213-226, (2016).
[7] H.Arzeli`es, Relativistic Kinematics, Pergamon, New York, pp. 173–180, 198, 201–203 (1966).
[8] H.Goldstein, C. Poole, and J. Safko, Classical Mechanics, Addison-Wesley, New York, 3rd ed., pp. 282–285 (2002).
[9] R.A.Muller, “Thomas precession: Where is the torque,” Am. J. Phys. 60, 313–317 (1992)
[10] W.Riddler, Relativity: Special, General and Cosmological, Oxford Univ. Press, Oxford, (2001).