Gravitational Field Wave as a Longitudinal Scalar Component of the Electromagnetic Field Waves

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Abstract

In this work it is found that the Einstein- Maxwell field equations for charge - free space admit plane - wave solutions that may be considered not only as electromagnetic (EM) waves, but also as gravitational (G) waves. This conclusion may be sustained by means of some local gauge transformation that leaves the energy-momentum tensor unchanged and introduces the G interaction into the classical electrodynamics. The result of this local gauge invariance is deriving of four field equations which turn out to be a more general form of Maxwell's equations. On the basis of these quations, it is shown that the plane EM and G waves are combined as a single EM wave in which the G field is proportional to the scalar longitudinal component of the EM wave, i.e. proportional to the scalar projection of electric field onto the common direction of propagation of the EM and G waves. In addition, starting from the principle of least action, the relativistic equation of a charged test particle in the field of EM and G waves is determined. In the non - relativistic limit, it is found that the parametric equations of test particle describe a curve similar to the so - called trochoid. Then, it is deduced that the motion of a test particle can not occurs under the influence of G field alone, since the inertial mass of test particle is related to the absolute value of its electric charge so that a test particle without electric charge must be a particle without inertial (rest) mass. These results suggest that the new derived form of the Maxwell's equations in free space may be proposed as a completion to the Einstein - Maxwell equations in free space.

Keywords : *orthogonal transformations, EM invariants, longitudinal vector component of the electromagnetic field waves, electro - gravito - magnetic (EGM) field, Weyl - like equation for describing massless particles*

I. Introduction

According to the principle of equivalence, a non - inertial reference system is equivalent to a certain G field. In what follows, we assume that the source of the "actual" G fields is the fundamental physical process of emission of EM radiation by accelerated charged particles that make up the matter. However, we limit our considerations only to the EM fields occurring in vacuum in the absence of electric charges. As we know, such EM fields are the EM waves described by the Maxwell equations in vacuum. Therefore, according to the principles of general relativity, we consider that the equations which describe the "actual" G fields are the Einstein - Maxwell equations for charge - free space, together with the Maxwell's equations for charge - free space, i.e.

$$\begin{gathered} R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -\frac{8\pi G_N}{c^4} T_{\alpha\beta} \\ \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 \\ \partial_\alpha F_{\alpha\beta} = 0 \end{gathered} \qquad (1)$$

where $T_{\alpha\beta}$ is the energy - momentum tensor of the EM field occurring in vacuum in the absence of electric charges, and $F_{\alpha\beta}$ is the EM field tensor.

II. Einstein - Maxwell Field Equations in Vacuum

Let us introduce a space - time equipped with a Riemannian metric, which is conformal to the pseudo - Euclidean space

$$\begin{cases} ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \psi^2 ds_E^2 \\ \quad ds_E^2 = \delta_{\alpha\beta} dx^\alpha dx^\beta \end{cases} \qquad (2)$$

that is we consider a local rescaling of the Euclidean metric tensor $\delta_{\alpha\beta} \to \psi^2(x^\mu)\delta_{\alpha\beta}$, where $\psi(x^\mu)$ is an arbitrary function of the space - time coordinates $x^\mu = (x, y, z, ict)$ and $\delta_{\alpha\beta}$ is the Kronecker delta tensor. The curvature tensor of this conformally flat space - time is the Riemann - Christoffel tensor

$$R^\alpha_{\beta\gamma\delta} = \partial_\delta \begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} - \partial_\gamma \begin{Bmatrix} \alpha \\ \beta\ \delta \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \sigma\ \gamma \end{Bmatrix} \begin{Bmatrix} \sigma \\ \beta\ \delta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \sigma\ \delta \end{Bmatrix} \begin{Bmatrix} \sigma \\ \beta\ \gamma \end{Bmatrix} \qquad (3)$$

where $\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix}$ represents the usual Christoffel connection. Since the metric tensor has the form $g_{\alpha\beta} = \psi^2(x^\mu)\delta_{\alpha\beta}$, the Christoffel connection can be rewritten as a Weyl conformal connection in the pseudo - Euclidean space

$$\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\} = \delta^{\alpha}_{\beta}\psi_{\gamma} + \delta^{\alpha}_{\gamma}\psi_{\beta} - \delta_{\beta\gamma}\psi^{\alpha} \quad (4$$

where the covariant and contravariant four - vectors are given by the expressions

$$\psi_{\beta} = \partial_{\beta} \ln \psi \quad (5)$$

$$\psi^{\alpha} = \delta^{\alpha\beta}\psi_{\beta} = \psi_{\alpha} \quad (6)$$

Inserting (4) into (3), and contracting α with γ, we can write the Einstein - Maxwell field equations in the form

$$\frac{1}{\psi}\partial_{\alpha}\partial_{\beta}\psi - \delta_{\alpha\beta}\left[\frac{1}{\psi}\square\psi - \frac{3}{2}\frac{1}{\psi^2}(\nabla\psi)^2\right] = -\frac{4\pi G_N}{c^4}T_{\alpha\beta}$$
(7)

where $\square$ denotes the d'Alembertian operator in pseudo - Euclidean space

$$\square\psi = \partial^{\alpha}\partial_{\alpha}\psi = \Delta\psi - (1/c^2)\partial_t^2\psi \quad (8)$$

$(\nabla\psi)^2$ denotes the expression

$$(\nabla\psi)^2 = \delta^{\alpha\beta}\partial_{\alpha}\psi\,\partial_{\beta}\psi \quad (9)$$

and $T_{\alpha\beta}$ is the energy - momentum tensor of the electromagnetic field, written in curvilinear coordinates (Gaussian system of units)

$$T_{\alpha\beta} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_{\beta}^{\ \gamma} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right) \quad (10)$$

These equations allow us to determine the unknown function $\psi(x^{\alpha})$ for a certain particular arrangement of the energy - momentum tensor $T_{\alpha\beta}$ of the electromagnetic field.

III. Electromagnetic Waves

Further on, we try to find the expression of the energy - momentum tensor for a plane EM wave. As we already know, the electromagnetic waves verify the Maxwell's equations in the absence of electric charges

$$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \quad ; \quad \mathrm{div}.\vec{H} = 0 \quad (11)$$

$$\nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \quad ; \quad \mathrm{div}.\vec{E} = 0 \quad (12)$$

Therefore, the electric and magnetic fields satisfy the same d'Alembert equation

$$\square\vec{E} = 0 \quad ; \quad \square\vec{H} = 0 \quad (13)$$

The simplest solutions to these equations are the plane - wave solutions of the form

$$\begin{cases} \vec{E}(\vec{r},t) = \vec{E}_0 e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em})} \\ \vec{E}_0 = E_{0x}\vec{i} + E_{0y}\vec{j} + E_{0z}\vec{k} \end{cases} \quad (14)$$

and, respectively

$$\begin{cases} \vec{H}(\vec{r},t) = \vec{H}_0 e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em})} \\ \vec{H}_0 = H_{0x}\vec{i} + H_{0y}\vec{j} + H_{0z}\vec{k} \end{cases} \quad (15)$$

where $\vec{E}_0$ and $\vec{H}_0$ are constant vectors, and α_{0em} is a constant (real) initial phase of the electromagnetic wave.

IV. The Invariants of the Electromagnetic Field

As it is already known, the electric and magnetic fields of the plane electromagnetic wave are perpendicular to each other and equal to each other in absolute value

$$\vec{E}\vec{H} = 0 \quad (16)$$

$$|\vec{E}| = E(\vec{r},t) = |\vec{H}| = H(\vec{r},t) \quad (17)$$

We also know that, the quantities $\vec{E}^2 - \vec{H}^2$ and $\vec{E}\vec{H}$ are fundamental invariants of the EM field, which remain unchanged with respect to the Lorentz transformations, but also with respect to the set of all possible rotations of the Cartesian coordinate systems. Therefore, the conditions (16) and (17) remain unchanged in the transition from one Cartesian coordinate system to another, because we are in the case where the two fundamental invariants are zero (null electromagnetic field).

V. Orthogonal Transformations

Excepting the Lorentz transformations, we want to introduce those linear transformations which preserves the absolute values of vectors and angles between them, i.e. those linear transformations that leaves unchanged the two fundamental invariants of the electromagnetic field. From linear algebra, we also know that such linear transformations are the so called
orthogonal transformations. Therefore, let us introduce a plane electromagnetic wave and two Cartesian coordinate systems, $S(0,x,y,z)$ and $S'(O,x',y',z')$, with the same origin. Then, let us consider that, with respect to the S' system, the electric $\vec{E}$ and magnetic $\vec{H}$ fields are constrained to oscillate in the y' and z' directions, respectively. This is what we call a linearly polarized plane wave traveling in the positive direction of the x' axis at the speed of light. Thus, we can write the electric and magnetic fields with respect to the two Cartesian coordinate systems, as follows

$$\vec{E}(\vec{r},t) = E(\vec{r},t)\vec{j'} = E_x(\vec{r},t)\vec{i} + E_y(\vec{r},t)\vec{j} + E_z(\vec{r},t)\vec{k} \qquad (18)$$

$$\vec{H}(\vec{r},t) = H(\vec{r},t)\vec{k'} = H_x(\vec{r},t)\vec{i} + H_y(\vec{r},t)\vec{j} + H_z(\vec{r},t)\vec{k} \qquad (19)$$

where $(\vec{i}, \vec{j}, \vec{k})$ are the orthonormal vectors of the coordinate system S, and $(\vec{i'}, \vec{j'}, \vec{k'})$ are the orthonormal vectors of the coordinate system S'. In what follows, we shall denote the orthonormal basis of the S' system $(\vec{i'}, \vec{j'}, \vec{k'})$ by $(\vec{n}, \vec{e}, \vec{h})$. Then, to write the orthonormal basis $(\vec{i'}, \vec{j'}, \vec{k'})$ in terms of $(\vec{i}, \vec{j}, \vec{k})$, we use the linear transformation

$$\begin{aligned} \vec{i'} &= \vec{n} = n_x\vec{i} + n_y\vec{j} + n_z\vec{k} \\ \vec{j'} &= \vec{e} = e_x\vec{i} + e_y\vec{j} + e_z\vec{k} \\ \vec{k'} &= \vec{h} = h_x\vec{i} + h_y\vec{j} + h_z\vec{k} \end{aligned} \qquad (20)$$

Now, it is easy to observe that the transformation matrix from S to S' can be displayed as

$$A = \begin{pmatrix} n_x & n_y & n_z \\ e_x & e_y & e_z \\ h_x & h_y & h_z \end{pmatrix} \qquad (21)$$

where the entries are the direction cosines of the transformation. It is known that the matrix representation of an orthogonal transformation is an orthogonal matrix whose rows and columns are mutually orthonormal vectors. This matrix verifies the matrix equation

$$A^T A = A\,A^T = I_3 \qquad (22)$$

where A^T is the transpose of matrix A, and I_3 is the identity matrix. From the matrix product $A\,A^T = I_3$, we get the well - known set of identities

$$\vec{n}^2 = n_x^2 + n_y^2 + n_z^2 = 1 \qquad (23)$$

$$\vec{e}^2 = e_x^2 + e_y^2 + e_z^2 = 1 \qquad (24)$$

$$\vec{h}^2 = h_x^2 + h_y^2 + h_z^2 = 1 \qquad (25)$$

$$\vec{n}\vec{e} = n_x e_x + n_y e_y + n_z e_z = 0 \qquad (26)$$

$$\vec{n}\vec{h} = n_x h_x + n_y h_y + n_z h_z = 0 \qquad (27)$$

$$\vec{h}\vec{e} = h_x e_x + h_y e_y + h_z e_z = 0 \qquad (28)$$

and from the matrix product $A^T A = I_3$, we get another set of identities

$$n_x^2 + e_x^2 + h_x^2 = 1 \qquad (29)$$

$$n_y^2 + e_y^2 + h_y^2 = 1 \qquad (30)$$

$$n_z^2 + e_z^2 + h_z^2 = 1 \qquad (31)$$

$$n_x n_y + e_x e_y + h_x h_y = 0 \qquad (32)$$

$$n_x n_z + e_x e_z + h_x h_z = 0 \qquad (33)$$

$$n_y n_z + e_y e_z + h_y h_z = 0 \qquad (34)$$

To these identities we add the cross product $\vec{n} = \vec{e} \times \vec{h}$, written in terms of components

$$n_x = e_y h_z - e_z \qquad (35)$$

$$n_y = e_z h_x - e_x h_z \qquad (36)$$

$$n_z = e_x h_y - e_y h_x \qquad (37)$$

VI. The Energy - Momentum Tensor

Let us now express the components of the energy - momentum tensor $T_{\alpha\beta}$ in terms of the electromagnetic field strength tensor $F_{\mu\nu}$. In matrix form, this tensor may be displayed as

$$F_{\mu\nu} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix} \qquad (38)$$

According to (18), (19) and (20) the components of electric and magnetic field strengths can be written as follows

$$\begin{aligned} E_x(\vec{r},t) &= E(\vec{r},t)e_x \\ E_y(\vec{r},t) &= E(\vec{r},t)e_y \\ E_z(\vec{r},t) &= E(\vec{r},t)e_z \end{aligned} \qquad (39)$$

and, respectively

$$\begin{aligned} H_x(\vec{r},t) &= H(\vec{r},t)h_x \\ H_y(\vec{r},t) &= H(\vec{r},t)h_y \\ H_z(\vec{r},t) &= H(\vec{r},t)h_z \end{aligned} \qquad (40)$$

where the absolute values of the electric and magnetic fields can be written as

$$|\vec{E}| = \sqrt{\vec{E}^2(\vec{r},t)} = E_0 e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em})} \qquad (41)$$

$$|\vec{H}| = \sqrt{\vec{H}^2(\vec{r},t)} = H_0 e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em})} \qquad (42)$$

In a similar manner, the components of the wave vector $\vec{K}$ can be written as follows

$$K_x = Kn_x,\ K_y = Kn_y,\ K_z = Kn_z \qquad (43)$$

Now, let us turn to the definition (10) of the energy - momentum tensor. By using the entries of the matrix (38), according to (17), we get

$$F_{\mu\nu}^2 = 2\left(\vec{H}^2 - \vec{E}^2\right) = 0 \qquad (44)$$

Therefore, the energy - momentum tensor given by (10) reduces to

$$T_{\alpha\beta} = \frac{1}{4\pi}\psi^{-2}F_{\alpha\gamma}F_{\beta\gamma} \qquad (45)$$

If we now calculate the component $T_{\alpha\beta}$, according to the identities (23) - (37) and the components of the electric and magnetic field strengths, (39) and (40), we can easily verify that the energy - momentum tensor can be written as

$$T_{\alpha\beta} = \frac{1}{4\pi}\psi^{-2}E^2(\vec{r},t)\frac{K_\alpha}{K}\frac{K_\beta}{K} \qquad (46)$$

where K_α is the well known four - dimensional wave vector $K_\alpha = \left(K_x, K_y, K_z, iK\right)$ of the electromagnetic wave. Substituting here the energy density of the electromagnetic field

$$w(\vec{r},t) = \frac{1}{8\pi}\left[E^2(\vec{r},t) + H^2(\vec{r},t)\right] = \frac{1}{4\pi}E^2(\vec{r},t) \qquad (47)$$

we can rewrite the energy - momentum tensor (46) in the final form

$$T_{\alpha\beta} = \psi^{-2}(\vec{r},t)w(\vec{r},t)\frac{K_\alpha}{K}\frac{K_\beta}{K} \qquad (48)$$

VII. Plane Wave – Like Disturbances in Space – Time

Substituting now (48) into (7), the Einstein - Maxwell vacuum equations become

$$\frac{1}{\psi}\partial_\alpha\partial_\beta\psi - \delta_{\alpha\beta}\left[\frac{1}{\psi}\Box\psi - \frac{3}{2}\frac{1}{\psi^2}(\nabla\psi)^2\right] = -\frac{4\pi G_N}{c^4}\frac{1}{\psi^2}\,w(\vec{r},t)\frac{K_\alpha}{K}\frac{K_\beta}{K} \qquad (49)$$

It is easy to show now that, using the four - dimensional notation, these equations admit plane - wave solutions of the form

$$\psi(x^\nu) = \psi_0 e^{i\left(x^\nu \tilde{K}_\nu - \alpha_{0g}\right)} \qquad (50)$$

where ψ_0 is a dimensionless real constant, α_{0g} is a constant (real) initial phase, and $\tilde{K}_\nu$ is the four - dimensional wave vector of this plane wave $\tilde{K}_\nu = \left(\tilde{K}_x, \tilde{K}_y, \tilde{K}_z, i\tilde{K}\right)$. In terms of it we find the following identities

$$\tilde{K}_\alpha^2 = \delta^{\alpha\beta}\tilde{K}_\alpha\tilde{K}_\beta = 0 \qquad (51)$$

$$\psi^\alpha = \psi_\alpha = \frac{1}{\psi}\partial_\alpha\psi = i\tilde{K}_\alpha \qquad (52)$$

$$\frac{1}{\psi}\partial_\alpha\partial_\beta\psi = -\tilde{K}_\alpha\tilde{K}_\beta \qquad (53)$$

These identities lead us to an eikonal equation and a wave equation

$$(\nabla\psi)^2 = \delta^{\alpha\beta}\partial_\alpha\psi \quad \partial_\beta\psi = 0 \qquad (54)$$

$$\Box\psi = \delta^{\alpha\beta}\partial_\alpha\partial_\beta\psi = 0 \qquad (55)$$

Therefore, the Einstein - Maxwell vacuum equations (49) reduce to

$$\tilde{K}_\alpha\tilde{K}_\beta = \frac{4\pi G_N}{c^4}\psi^{-2}(\vec{r},t)w(\vec{r},t)\frac{c^2}{\omega_0^2}K_\alpha K_\beta \qquad (56)$$

VIII. The Relationship between Amplitudes

Since the wave function $\psi(x^\nu)$ describes a plane - wave disturbance in pseudo - Euclidean space, we may consider that ψ_0 represents the amplitude of the plane - wave disturbance. It is easy to show that the wave vector $\vec{\tilde{K}}$ is the same wave vector $\vec{K}$ of the plane electromagnetic wave. Indeed, substituting (41) into (47), the energy density of the EM field becomes

$$w(\vec{r},t) = \frac{1}{4\pi}E_0^2 e^{2i\left(x^\nu K_\nu - \alpha_{0em}\right)} \qquad (57)$$

Now, substituting (50) and (57) into (56), the Einstein - Maxwell equations become

$$\tilde{K}_\alpha\tilde{K}_\beta e^{2i\left(x^\nu\tilde{K}_\nu - \alpha_{0g}\right)} = \frac{G_N}{c^2\omega_0^2}E_0^2\psi_0^{-2}e^{2i\left(x^\nu K_\nu - \alpha_{0em}\right)}K_\alpha K_\beta \qquad (58)$$

It follows that $\tilde{K}_\alpha = K_\alpha$, i.e. the EM waves and the space - time disturbances have the same angular frequency, $\tilde{\omega}_0 = \omega_0$, and the same propagation direction, $\vec{\tilde{K}} = \vec{K}$, and the relationship between the electromagnetic wave amplitude and space - time wave amplitude is given by

$$\psi_0^2 = \frac{G_N}{c^2\omega_0^2}E_0^2 e^{2i\left(\alpha_{0g} - \alpha_{0em}\right)} \qquad (59)$$

Also, the energy - momentum tensor (48) becomes independent of time and of position

$$T_{\alpha\beta} = \frac{1}{4\pi}\psi_0^{-2}E_0^2\frac{K_\alpha}{K}\frac{K_\beta}{K}e^{2i\left(\alpha_{0g} - \alpha_{0em}\right)} \qquad (60)$$

IX. The Conservation Law

We know that the Einstein - Maxwell equations in vacuum ensure the fact that the covariant divergence of the energy - momentum tensor is zero. Though the equation

$$\nabla_\alpha T^\alpha{}_\beta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}T^\alpha{}_\beta\right) - T^\alpha{}_\gamma\begin{Bmatrix}\gamma \\ \alpha\ \beta\end{Bmatrix} = 0 \qquad (61)$$

does not express the conservation law of energy and momentum, howbeit the conservation law is ensured. Indeed, substituting (52) into (4), the Christoffel connection can be written as

$$\left\{ {\gamma \atop \alpha\ \beta} \right\} = i\left(K_\alpha \delta_\beta^\gamma + K_\beta \delta_\alpha^\gamma - K^\gamma \delta_{\alpha\beta}\right) \qquad (62)$$

Now, using (51) and (60), we can easily verify that $T^\alpha{}_\gamma \left\{ {\gamma \atop \alpha\ \beta} \right\} = 0$. Therefore, (61) becomes

$$\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}T^\alpha{}_\beta\right) = 0 \qquad (63)$$

that ensures the conservation law of energy and momentum of the electromagnetic field. In what follows, we shall see that the constant tensor (60) may be interpreted as energy - momentum tensor not only of the electromagnetic field, but also of the gravitational field.

X. The Equation of Motion

Let us now find the motion equation of a test particle in conformally flat space - time (2). According to the principle of least action from the relativistic mechanics, the Lagrangian for a free material particle is given by

$$L_0(\vec r, \vec v, t) = -m_0 c^2 \psi(\vec r, t)\sqrt{1 - \frac{\vec v^2}{c^2}} \qquad (64)$$

where m_0 is the rest mass of the particle. This is the Lagrangian for uncharged test particles. For the charged test particles we must add the terms $(q/c)\vec A \vec v - q\varphi(\vec r, t)$ which describe the interaction of the charge with the electromagnetic field. Thus, the Lagrangian of a charged test particle is given by

$$L(\vec r, \vec v, t) = -m_0 c^2 \psi(\vec r, t)\sqrt{1 - \frac{\vec v^2}{c^2}} + \frac{q}{c}\vec A \vec v - q\varphi(\vec r, t) \qquad (65)$$

Substituting this Lagrangian into the Euler – Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec v}\right) - \frac{\partial L}{\partial \vec r} = 0 \qquad (66)$$

for a plane "gravitational" wave having the space - time dependence given by (50), we get the following relativistic equation of motion

$$\frac{d\vec p}{dt} + i\left(\vec K \vec v - \omega_0\right)\vec p + i m_0 c^2 \vec K\sqrt{1 - \frac{\vec v^2}{c^2}} = \psi^{-1}\left[q\vec E + \quad + \frac{q}{c}\left(\vec v \times \vec H\right)\right]$$
(67)

where $\vec p$ represents the relativistic momentum. For an EM wave having the space - time dependence given by (14) and (15), and for $\alpha_{oem} = \alpha_{og}$, the motion equation can be rewritten as

$$\frac{d\vec p}{dt} - i\omega_0\left(1 - \frac{\vec n \vec v}{c}\right)\vec p + i m_0 c\omega_0 \vec n\sqrt{1 - \frac{\vec v^2}{c^2}} = \\ = \frac{q}{\psi_0}\left[\vec E_0 + \frac{1}{c}\left(\vec v \times \vec H_0\right)\right]$$
(68)

where, as mentioned above, we have

$$\vec e = \left(e_x, e_y, e_z\right) \quad ; \quad \vec h = \left(h_x, h_y, h_z\right) \\ \vec E_0 = E_0 \vec e \quad ; \quad \vec H_0 = H_0 \vec h \\ \vec n = \left(n_x, n_y, n_z\right) \ ; \ \vec v = (\dot x, \dot y, \dot z)$$
(69)

The equation (68) represents the equation of motion of a charged test particle in static uniform electromagnetic and "gravitational" fields. Though the gravitational field strength does not occur explicitly, in what follows we shall see that it is possible to define a scalar gravitational field whose intensity is proportional to the amplitude ψ_0 of the space - time wave. Now, since for a plane electromagnetic wave, we have the cross product $\vec H_0 = \vec n \times \vec E_0$, the right - hand side of motion equation can be rearranged as

$$\frac{d\vec p}{dt} - i\omega_0\left(1 - \frac{\vec n \vec v}{c}\right)\vec p + i m_0 c\omega_0 \vec n\sqrt{1 - \frac{\vec v^2}{c^2}} = \\ \frac{q}{\psi_0}\left[\vec E_0\left(1 - \quad - \frac{\vec n \vec v}{c}\right) + \vec n\left(\frac{\vec v}{c}\vec E_0\right)\right]$$
(70)

In the weak electromagnetic field approximation, we may limit ourselves to the case where the velocity of the charge $v \ll c$. Then, for simplicity, we may choose to solve the equation with respect to the coordinate system S'. Hence, in the non - relativistic limit, when we have $1 - \vec v^2/c^2 \cong 1$, and $1 - \vec n\, \vec v/c \cong 1$, the equation of motion with respect to the coordinate system S' goes over into the form

$$\frac{d\vec v'}{dt} - i\omega_0 \vec v' + i\omega_0 c\, \vec n = \frac{q}{m_0}\psi_0^{-1}\left[\vec E_0 + \vec n\left(\frac{\vec v'}{c}\vec E_0\right)\right]$$
(71)

where the vectors have the components

$$\vec E_0 = -\vec n \times \vec H_0 = (0, E_0, 0) \qquad (72)$$

$$\vec n = \vec i = (1, 0, 0) \ ; \ \vec v' = (\dot x', \dot y', \dot z') \qquad (73)$$

Therefore, expressed in terms of components, the equation can be written as

$$\ddot x' - i\omega_0 \dot x' + i\omega_0 c - \sigma E_0 \dot y' = 0 \qquad (74)$$

$$\ddot y' - i\omega_0 \dot y' - \sigma_0 E_0 = 0 \qquad (75)$$

$$\ddot{z}' - i\omega_0 \dot{z}' = 0 \quad (76)$$

where we have introduced the notations

$$\sigma_0 = \frac{q}{m_0}\psi_0^{-1}, \quad \sigma = \frac{\sigma_0}{c} \quad (77)$$

The third equation can be immediately integrated to give

$$\dot{z}' = C_z e^{i\omega_0 t} \quad (78)$$

where C_z is a complex constant. Also, the second equation can be easily integrated to give

$$\dot{y}' = iC_y e^{i\omega_0 t} + i\frac{\sigma E_0}{K} \quad (79)$$

where C_y is another complex constant. By substituting this solution into (74), we get

$$\ddot{x}' - i\omega_0 \dot{x}' + i\omega_0 c - ia_1 e^{i\omega_0 t} - ia_2 = 0 \quad (80)$$

where we have introduced the notations

$$\begin{cases} a_1 = \sigma E_0 C_y \\ a_2 = \frac{(\sigma E_0)^2}{K} \end{cases} \quad (81)$$

The integral of this equation is

$$\dot{x}' = (C_x + ia_1 t)e^{i\omega_0 t} + c\left(1 - \frac{a_2}{c\omega_0}\right) \quad (82)$$

where C_x is a complex constant. Since the complex constants C_x, C_y, C_z are multiplied by $e^{i\omega_0 t}$, we can, by a suitable choise of the time origin, give them real values. Thus, by extracting the real parts from the above complex expressions, we obtain

$$\dot{x}' = C_x \cos\omega_0 t - a_1 t \sin\omega_0 t + c\left(1 - \frac{a_2}{c\omega_0}\right)$$
(83)

$$\dot{y}' = -C_y \sin\omega_0 t \quad (84)$$

$$\dot{z}' = C_z \cos\omega_0 t \quad (85)$$

Therefore, the velocity components are periodic functions of the time, and at $t = 0$ the velocity $\vec{v}_0'$ is contained in the (x', z') plane

$$\vec{v}_0' = \dot{x}_0' \vec{n} + \dot{z}_0' \vec{h} \quad (86)$$

where the components are given by

$$\dot{x}_0' = C_x + c\left(1 - \frac{a_2}{c\omega_0}\right)$$
$$\dot{z}_0' = C_z$$

Now, by using the well - known formula

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)dt$$

where $= 2\pi/\omega_0$, we find the average values

$$\langle \dot{x}' \rangle = c\left(1 - \frac{a_3}{c\omega_0}\right)$$
$$\langle \dot{y}' \rangle = 0$$
(87)
$$\langle \dot{z}' \rangle = 0$$

where we have $a_3 = a_2 - a_1$. So, the average velocity $\langle v' \rangle$ is along the x' axis, i.e. it is perpendicular to the electric and magnetic fields and independent of the electric charge sign. For the average velocity to be small compared with the velocity of light, i.e. $\langle \dot{x}' \rangle \ll c$, we must have

$$1 - a^2 + a\frac{C_y}{c} \ll 1 \quad (88)$$

where we have $a = \frac{\sigma_0 E_0}{\omega_0 c} = \frac{qE_0}{m_0 c}\frac{\psi_0^{-1}}{\omega_0}$. Thus, if we note $q = \pm|q|$, we get $a = \pm\frac{\omega}{\omega_0}\psi_0^{-1}$, where we have

$$\omega = \frac{|q|E_0}{m_0 c} \quad i.e. \quad \omega = \frac{|q|H_0}{m_0 c} \quad (89)$$

From (88), we obtain : $a\frac{C_y}{c} \ll a^2$, where $a = \pm|a|$, $|a| = \frac{\omega}{\omega_0}\psi_0^{-1}$. Thus, we can write

$$\pm|a|\frac{C_y}{c} \ll |a|^2 \Leftrightarrow |a|\left|\frac{C_y}{c}\right| \ll |a|^2 \Leftrightarrow \left|\frac{C_y}{c}\right| \ll |a|$$
(90)

Since we must have $\left|\frac{C_y}{c}\right| \ll 1$, it follows that $|a| = 1 \Leftrightarrow \frac{\omega}{\omega_0}\psi_0^{-1} = 1$. Thus, we obtain

$$\psi_0 = \frac{\omega}{\omega_0} \quad (91)$$

So, for the average velocity to be small compared with the velocity of light, it is necessary that the electric and magnetic fields satisfy the relation

$$E_0 = H_0 = \frac{m_0}{|q|}c\omega_0\psi_0 \quad (92)$$

By integrating the equations (83) - (85) once more, and choosing the constants of integration so that at $t = 0$, $x'(0) = 0, y'(0) = 0, z'(0) = 0$, we get the coordinates of the charged test particle as periodic functions of the time

$$x'(t) = \frac{1}{\omega_0}\left(C_x - \frac{a_1}{\omega_0}\right)\sin\omega_0 t + \frac{a_1}{\omega_0} t\cos\omega_0 t + c\left(1 - \frac{a_2}{\omega_0 c}\right)t$$

$$y'(t) = \frac{C_y}{\omega_0}(\cos\omega_0 t - 1)$$
(93) – (95)

$$z'(t) = \frac{c_z}{\omega_0}\sin\omega_0 t$$

As we already know, the first two equations describe the projection of the particle motion on the (x', y') plane. We observe that this projection is a curve similar to the so - called trochoid. The third equation describes an oscillatory motion directed along the z' axis. The angular frequency of the oscillatory motion is the proper frequency of the electromagnetic field. Now, if we recall (59), for $\alpha_{oem} = \alpha_{og}$ (i.e. the EM waves and the space - time waves are in phase), we may write

$$E_0 = H_0 = \frac{c\omega_0}{\sqrt{G_N}}\psi_0 \qquad (96)$$

Therefore, comparing (96) and (92), we obtain

$$m_0 = \frac{1}{\sqrt{G_N}}|q| \qquad (97)$$

Thus, the inertial mass of a test particle is related to the absolute value of its electric charge, suggesting that the test particle possesses a kind of gravitational "charge" named gravitational mass or inertial mass, whose value is proportional to the absolute value of its electric charge. Therefore, it would be erroneous to consider the Lagrangian (64) as Lagrangian for uncharged test particles, i.e. it would be erroneous to consider the equation

$$\frac{d\vec{p}}{dt} - i\omega_0\left(1 - \vec{n}\frac{\vec{v}}{c}\right)\vec{p} + im_0 c\omega_0\vec{n}\sqrt{1 - \frac{\vec{v}^2}{c^2}} = 0$$
(98)

as equation of motion for uncharged test particles, since a test particle without electric charge would be a particle without rest mass, and a particle without rest mass would be a field particle, namely a boson (in our case, foton or graviton). Hence, the test particle verifies the generalized Lorentz force equation (68), since for an inertial mass m_0 it is necessary to be associate an electric charge q whose absolute value is given by the relation

$$|q| = \sqrt{G_N}m_0 \qquad (99)$$

XI. Electro – Gravito – Magnetic Wave

As a consequence of the Einstein - Maxwell field equations and of the relationships between the electromagnetic and space - time wave amplitudes given by (96), it is possible to associate with the space - time wave a gravitational wave whose amplitude is proportional to the amplitude ψ_0 of the space - time wave. Indeed, if we introduce the expression

$$w_0 = \frac{1}{8\pi}(E_0^2 + H_0^2) \qquad (100)$$

as energy density of a constant and uniform electromagnetic field, then, according to (96), we can rewrite this expression into the equivalent form

$$w_0 = \frac{1}{8\pi}\left(\sqrt{\frac{2}{G_N}}\Gamma_0\right)^2 \qquad (101)$$

where the quantity $\Gamma_0 = c\omega_0\psi_0 = c\omega$ may be considered as intensity of the G field. Hence, the expression (101) may also be considered as the energy density of a constant and uniform gravitational field. In a similar manner, we may rewrite the expression of the energy - momentum tensor (60) under the equivalent form

$$T_{\alpha\beta} = \frac{1}{8\pi}\psi_0^{-2}\left(\sqrt{\frac{2}{G_N}}\Gamma_0\right)^2\frac{K_\alpha}{K}\frac{K_\beta}{K} \qquad (102)$$

where, according to (96), we have used the relationships

$$\Gamma_0 = \sqrt{G_N}E_0 = \sqrt{G_N}H_0 \qquad (103)$$

Therefore, the energy - momentum tensor $T_{\alpha\beta}$ of the electromagnetic field may also be considered as energy - momentum tensor of the gravitational field. Now, let us multiply the relationships (103) by $e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0g})}$. Considering $\alpha_{0em} = \alpha_{0g}$, we can write

$$\Gamma(\vec{r}, t) = \Gamma_0 e^{-i(\omega_0 t - \vec{K}\vec{r} + \alpha_{0g})} \qquad (104)$$

where $\Gamma(\vec{r}, t) = \sqrt{G_N}E(\vec{r}, t) = \sqrt{G_N}H(\vec{r}, t)$ represents the scalar field of a plane gravitational wave. Further on, let us introduce a non - symmetric electromagnetic field tensor of the form

$$G_{\mu\nu} = F_{\mu\nu} + \Gamma_t\delta_{\mu\nu} \qquad (105)$$

or, explicitly, as a matrix, according to (38)

$$G_{\mu\nu} = \begin{pmatrix} \Gamma_t & H_z & -H_y & -iE_x \\ -H_z & \Gamma_t & H_x & -iE_y \\ H_y & -H_x & \Gamma_t & -iE_z \\ iE_x & iE_y & iE_z & \Gamma_t \end{pmatrix} \qquad (106)$$

where the component Γ_t is an unknown function of the space - time coordinates $x^\mu = (x, y, z, ict)$. It is easy to verify that the transformed energy - momentum tensor

$$T_{\alpha\beta}(G) = \frac{1}{4\pi}\psi^{-2}\left(G_{\alpha\gamma}G_{\beta\gamma} - \frac{1}{4}\delta_{\alpha\beta}G_{\mu\nu}^2\right)$$

is the same energy - momentum tensor $T_{\alpha\beta}(F)$ given by (10), i.e.

$$T_{\alpha\beta}(G) = T_{\alpha\beta}(F) \qquad (107)$$

Hence, the EM field tensor $F_{\mu\nu}$ is not uniquely determined for the energy - momentum tensor $T_{\alpha\beta}$. In addition, using the well - known covariant formulation of Maxwell's field equations in terms of the transformed electromagnetic tensor $G_{\mu\nu}$, i.e.

$$\partial_\lambda G_{\mu\nu} + \partial_\mu G_{\nu\lambda} + \partial_\nu G_{\lambda\mu} = 0 \qquad (108)$$

$$\partial_\nu G_{\nu\mu} = 0 \qquad (109)$$

we get the following set of four field equations for charge - free space

$$\begin{cases} \nabla \times \vec{H} = \frac{1}{c}\partial_t \vec{E} + \mathrm{grad.}\,\Gamma_t \\ \nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{H} \\ div.\vec{\mathrm{E}} + \frac{1}{\mathrm{c}}\partial_\mathrm{t}\Gamma_\mathrm{t} = 0 \\ div.\vec{H} = 0 \end{cases} \qquad (110)$$

It is easy to verify that these equations lead us to the wave equation for the electric, magnetic and Γ_t fields

$$\Box\vec{E} = 0,\ \ \Box\Gamma_t = 0,\ \ \Box\vec{H} = 0 \qquad (111)$$

Therefore, the local invariance (107) of the energy - momentum tensor $T_{\alpha\beta}$ introduces, in addition to the electromagnetic field, the scalar field Γ_t which verifies the same wave equation like as the EM field. Let us consider that these equations admit plane - wave solutions of the form

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{-i\left(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em}\right)} \qquad (112)$$

$$\vec{H}(\vec{r},t) = \vec{H}_0 e^{-i\left(\omega_0 t - \vec{K}\vec{r} + \alpha_{0em}\right)} \qquad (113)$$

$$\Gamma_t(\vec{r},t) = \Gamma_{0t} e^{-i\left(\omega_0 t - \vec{K}\vec{r} + \alpha_{0g}\right)} \qquad (114)$$

where α_{0g} is the initial phase of the field wave $\Gamma_t(\vec{r},t)$ and α_{0em} is the initial phase of the electromagnetic field wave. There is no loss of generality if we choose the versors and the vectors $\vec{K}$ and $\vec{H}$ as follows

$$\vec{n} = (1,0,0)$$
$$\vec{e} = (0,1,0) \qquad (115)$$
$$\vec{h} = (0,0,1) \quad \text{and}$$
$$\vec{K} = K\,\vec{n} \qquad (116)$$
$$\vec{H} = H\vec{h} \qquad (117)$$

Substituting the solutions (112) - (114) into the field equations (110), we find that the three components of the field are related by

$$\vec{E} = \vec{n}\,\Gamma_t - \vec{n} \times \vec{H} \qquad (118)$$
$$\vec{H} = \vec{n} \times \vec{E} \qquad (119)$$
$$\Gamma_t = \vec{n}\,\vec{E} \qquad (120)$$

From (120) we find a more general relation between $\vec{E}$ and $\vec{n}$. However, the relationship between $\vec{H}$ and $\vec{n}$ remains unchanged

$$\vec{n}\,\vec{H} = 0 \qquad (121)$$

Therefore, the field Γ_t represents the scalar projection of the electric field $\vec{E}$ onto the direction of propagation of the EM wave. By comparing the equations (120) and (114), we also find

$$\Gamma_{0t} = \vec{n}\,\vec{E}_0 = E_0 \cos\varphi \qquad (122)$$

where φ is the angle between $\vec{E}$ and $\vec{n}$, supposed to be different from $\pi/2$. By identification it is also found that $\alpha_{0em} = \alpha_{0g}$, i.e. the electric, magnetic and Γ_t field waves are in phase. Further on, according to (103) and (104), we can write

$$\Gamma_t = \mathrm{E}(\vec{r},t)\cos\varphi = \frac{\cos\,\varphi}{\sqrt{G_N}}\Gamma(\vec{r},t) \qquad (123)$$

i.e. the scalar component Γ_t of the EM tensor $G_{\mu\nu}$ is proportional to the G field strength $\Gamma(\vec{r},t)$. In addition, it is easy to observe that the electric field $\vec{E}$ is contained in the (x,y) plane, since $\vec{E}\,\vec{H} = 0$, and $\vec{H}$ is a normal vector to the (x,y) plane. Hence, the electric field can be decomposed into a sum of two orthogonal vector components : a parallel component $\vec{E}_{||} = \vec{E}_x$ and a perpendicular component $\vec{E}_\perp = \vec{E}_y$ to the direction of propagation of the wave. Thus, the first component becomes the longitudinal component of electric field wave, and the other becomes the transverse component of electric field wave. So, we can write

$$\vec{E} = \vec{E}_{long.} + \vec{E}_{transv.} \qquad (124)$$

where, according to (118), we find

$$\vec{E}_{long.} = \vec{E}_x = \vec{E}_{||} = \Gamma_t\,\vec{n} \qquad (125)$$
$$\vec{E}_{transv.} = \vec{E}_y = \vec{E}_\perp = -\,\vec{n} \times \vec{H} = H\vec{e} \qquad (126)$$

As a consequence, we find that the EM wave becomes not only a transverse wave but also a longitudinal wave, and the longitudinal scalar component of the wave is even the field Γ_t. Moreover, we shall see that the field Γ_t can be associated with the time direction. Thus, the three fields (112) - (114) may be considered as components of a single electromagnetic wave, namely the electro - gravito - magnetic field wave. The existence of this field can also be demonstrated by means of a wave equation similar to the Weyl equation for describing massless particles. Indeed, for such a particle we can use the relativistic wave equation proposed by Weyl

$$\sigma^\mu \partial_\mu \Psi = 0 \ \ (\mu = 1,2,3,4) \qquad (127)$$

where the matrices $\sigma^i = \sigma_i$ $(i = 1, 2, 3)$ are 4×4 Hermitian matrices that have the squares equal to the identity matrix I_4 and they all mutually anticommute

$$\begin{cases} \sigma_i\sigma_j + \sigma_j\sigma_i = 0, \;\; i \neq j \\ \sigma_i^2 = I_4 \end{cases} \tag{128}$$

and $\sigma^4 = \sigma_4 = iI_4$. Also, Ψ is the transpose of the usual wave function with four components $\Psi = (\Psi_x, \Psi_y, \Psi_z, \Psi_t)^T$. Explicitly, the set of three matrices σ_i may be displayed as

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_2 = \sigma_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{129}$$

$$\sigma_3 = \sigma_z = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

Substituting these matrices into (128), we get two separated equations

$$\begin{cases} \nabla \times \vec{\Psi} = \frac{i}{c}\partial_t\vec{\Psi} - grad.\Psi_t \\ div.\vec{\Psi} = -\frac{i}{c}\partial_t\Psi_t \end{cases} \tag{130}$$

Now it is easy to observe that, if we identify $\vec{\Psi}$ and Ψ_t as

$$\begin{cases} \vec{\Psi} = \vec{E} + i\vec{H} \\ \Psi_t = -i\Gamma_t \end{cases} \tag{131}$$

then the wave equations (130) go over into the electro - gravito - magnetic field equations (110). Also, we can see that, the four - component wave function consists of three fields, two vector fields (the electric and magnetic fields) and a scalar field (the Γ_t field) that is proportional to the gravitational field strength Γ. Then, the time component of the wave function is given by the component Γ_t of $G_{\mu\nu}$ tensor, multiplied by $-i$. This means that both the component Γ_t of $G_{\mu\nu}$ tensor and the scalar gravitational field strength Γ are associated with the time direction.

1. Conclusions

Usually, in general relativity theory, the dimensionless quantity $\psi^2(\vec{r}, t)$ is associated with the scalar gravitational potential $V_g(\vec{r}, t)$, according to the formula

$$\psi^2(\vec{r}, t) = \frac{1}{c^2}V_g(\vec{r}, t)$$

In the present paper we have associated the same quantity with the square of gravitational field strength $\Gamma(\vec{r}, t)$, according to the formula

$$\psi^2(\vec{r}, t) = \frac{1}{c^2\omega_0^2}\Gamma^2(\vec{r}, t)$$

From this association we have defined the amplitude of gravitational field wave as $\Gamma_0 = c\omega_0\psi_0$, where ψ_0 is the amplitude of space - time wave. Then, we have found that the energy - momentum tensor $T_{\alpha\beta}$ is an invariant with respect to the local gauge transformation

$$F_{\alpha\beta}(x^\nu) \rightarrow F_{\alpha\beta}(x^\nu) + \Gamma_t(x^\nu)\delta_{\alpha\beta}$$

This transformation introduces the G interaction into the Maxwell equations, setting out a connection between the EM and G fields in addition to the connection established by the Einstein - Maxwell equations. From here, we have deduced the existence of EGM wave which is a wave of oscillating electric, magnetic and Γ_t fields. In other words, the G wave has become the third component of an EM wave whose relation between the electric field vector $\vec{E}$ and the wave vector $\vec{K}$ is more general than that resulted from the Maxwell's equations. Thus, the EGM field wave turned out to be an EM wave which consist of two transverse vector components, $\vec{E}_\perp$ and $\vec{H}$, and a longitudinal vector component, $\vec{E}_{||}$. It is worth noting that, according to (116) and (126), the absolute value of magnetic field, H, occurs both as the component of magnetic field along the z axis, $H_z = H$, and as the scalar projection of electric field onto the y axis, as if the electric field of EM wave would be converted into another magnetic field, perpendicular to the first. Also, it is worth noting that the difference between the EGM field equations and Einstein - Maxwell field equations can also be characterized by means of the EM invariants. Indeed, multiplying the equations (118) and (119) by $\vec{E}$, we obtain the relations

$$\begin{cases} \vec{E}^2 = \vec{H}^2 + \Gamma_t^2 \\ \vec{E}\,\vec{H} = 0 \end{cases}$$

which, in terms of electromagnetic invariants, can be written as

$$\begin{cases} \vec{E}^2 - \vec{H}^2 = \Gamma_t^2 = inv. \\ \vec{E}\,\vec{H} = 0 = inv. \end{cases} \quad i.e. \quad \begin{cases} E > H \\ \vec{E}\,\vec{H} = 0 \end{cases}$$

Then, the Einstein - Maxwell field equations for charge - free space determine null EM fields, that is EM fields which verify the conditions

$$\begin{cases} \vec{E}^2 - \vec{H}^2 = 0 = inv. \\ \vec{E}\,\vec{H} = 0 = inv. \end{cases} \quad i.e. \quad \begin{cases} E = H = \left(1/\sqrt{G_N}\right)\Gamma \\ \vec{E}\,\vec{H} = 0 \end{cases}$$

Therefore, it is easy now to observe that the EGM field equations (110) correspond to the Einstein - Maxwell equations for the case where $E > H$.

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