Classical and Quantum Motion of Particles in Rindler Space

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Abstract

In this article we have investigated both classical motion and quantum mechanical motion of a particle in the frame undergoing a uniform accelerated motion. We have also studied the Bohr-Sommerfeld quantization problem in Rindler space and obtained the quantized energy eigen values. We have noticed that the quantized system behaves like a wave packet with the quadratic form of quantized energy eigenspectrum. We have further obtained the energy eigen values in Rindler space using Dirac notation for the eigen states of the particle by introducing creation and annihilation operators. In this case we have solved the energy eigen value problem using the first order time independent perturbation theory. For the sake of completeness, we have also investigated the motion of massless particles. It should be noted that in this article we have used the non-relativistic form of Rindler Hamiltonian.

Keywords - *Rindler space,Quantum mechanics,Uniformly accelerated frame,Unruh temperature,Hawking radiation.*

I. INTRODUCTION

It is well known that the conventional Lorentz transformations are the space-time coordinate transformations between two inertial frame of references [1]. Now following the principle of equivalence, it is trivial to obtain the space-time transformations between a uniformly accelerated frame and an inertial frame and vice-versa with the same mathematical formalism as it is done in special theory of relativity [2–6]. In the present scenario the space-time geometry can very easily be shown to be flat in nature, which is called the Rindler space. For the sake of illustration of principle of equivalence, we may state, that a reference frame undergoing an accelerated motion in absence of gravitational field is equivalent to a frame at rest in presence of a gravitational field. Therefore in the present picture, the magnitude of the uniform acceleration is exactly equal to the strength of gravitational field. However, the reverse is not always true. We may assume that the gravitational field is produced by a strong gravitating object, e.g., a stellar black hole. We further approximate that the gravitational field is constant within a small domain of spatial region. Since it is exactly equal to the uniform acceleration of the moving frame, this is also called the local acceleration of the frame. We have arranged the article in the following manner: In the next section, we have given the basic formalism for the motion in Rindler space. In section 3, for the sake of completeness we have studied the classical motion of the particle in Rindler space. In section 4 we have developed a formalism for Bohr-Sommerfeld Quantization of Classical Motion of the Particle and obtained the energy eigen values. In section 5 we have obtained the Heisenberg Equation of Motion in Rindler Space. In section 6 using the Dirac abstract notation we have obtained the energy eigen vales in Rindler space using first order time independent perturbation theory. In section 7, the energy eigen values and the corresponding eigen states are obtained in Rindler space for massless particle and finally in section 8 we have given the conclusion. To the best of our knowledge, such studies have not been reported earlier.

II. BASIC FORMALISM

In this section, for the sake of completeness, following the references [7–9] we have established some of the useful formulas of special theory of relativity for a uniformly accelerated frame of reference. Before we go to the scenario of uniform acceleration of the moving frame, let us first assume that the frame S′ has rectilinear motion with uniform velocity v along x-direction with respect to some inertial frame S. Further the coordinates of an event occurred at the point P (say) is indicated by (x,y,z,t) in S-frame and (x′,y′,z′,t′) in the frame S′. The primed and the un-primed coordinates are related by the conventional form of Lorentz transformations and are given by

$$x' = \Upsilon(x - vt),\ y' = y, z' = z \text{ and}$$
$$t' = \Upsilon(t - vx) \text{ where } \Upsilon = (1 - v^2)^{-1/2} \quad (1)$$

is the well known Lorentz factor. Next we consider a uniformly accelerated frame S′ moving with uniform acceleration α, which is also along x-direction relative to S-frame. Then the Rindler coordinates are given by (see the references [7–9], here we have considered the natural units, c=ħ=1),

$t = \left(\frac{1}{\alpha} + x'\right) \sinh(\alpha t')$ and

$x = \left(\frac{1}{\alpha} + t'\right) \cosh(\alpha t')$ (2)

Hence one can also express the inverse relations

$t' = \frac{1}{2\alpha} ln\left(\frac{x+t}{x-t}\right)$ and

$x' = (x^2 - t^2)^{1/2} - \frac{1}{\alpha}$ (3)

The Rindler space-time coordinates as mentioned above are just the accelerated frame transformations of the Minkowski metric of special relativity. The Rindler coordinate transformations change the Minkowski line element from

$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ to (4)

$ds'^2 = (1 + \alpha x')^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$ (5)

Since the motion is assumed to be rectilinear and along x-direction, dy'=dy and dz'=dz. The form of the metric tensor in 1+1-dimension can then be expressed as

$g^{\mu\nu} = diag((1 + \alpha x)^2, -1)$ (6)

Since we are dealing with accelerated frame only, hence forth the prime symbols are dropped. Now with the kinematics of calculation of particle motion in Minkowski space as discussed in [1], the action integral may be written as (see also [10] and [11])

$S = \alpha_0 \int_a^b ds \equiv \int_a^b L\, dt$ (7)

Then using eqns.(5) and (7) and putting $\alpha_0 = -m_0$[1], where m0 is the rest energy of the particle, the Lagrangian of the particle is given by [11]

$L = -m_0[(1 + \alpha x)^2 - v^2]^{1/2}$ (8)

where v is the three velocity of the particle. The momentum of the particle is then given by

$p = m_0 v[(1 + \alpha x)^2 - v^2]^{-1/2}$ (9)

Then from the definition, the Hamiltonian of the particle may be written as

$H = pv(p) - L$ or (10)

$H = \varepsilon(p) = m_0(1 + \alpha x)\left(1 + \frac{p^2}{m_0^2}\right)^{1/2}$ (11)

This is the well known Rindler Hamiltonian. Then it can very easily be shown that in the non-relativistic approximation, the Hamiltonian is given by

$H = (1 + \alpha x)\left(\frac{p^2}{2m_0} + m_0\right)$ (11a)

In the classical level, the quantities H, x and p are treated as dynamical variables. In the next section we shall investigate the classical motion of the particle in Rindler space.

III. CLASSICAL MOTION IN RINDLER SPACE

In this section we have investigated the time evolution for both the space and the momentum variables of the particle moving in Rindler space. We have considered both the relativistic as well as the non-relativistic form of the Rindler Hamiltonian (eqns.(11) and (11a) respectively). Hence we have also obtained the classical phase space trajectories for the particle in the Rindler space. We have noticed that in the relativistic scenario, both the spatial and the momentum coordinates are real in nature and diverge as $t \to \infty$. For both the variables the time dependencies are extremely simple. Hence we have obtained classical trajectories p(x) by eliminating the time dependent part. However, in the non-relativistic approximation, the spatial coordinates are quite complex in nature, whereas the momentum coordinates are purely imaginary. Since the mathematical form of the phase space trajectories are quite complicated, we have obtained p(x) numerically in the non-relativistic scenario.

A. RELATIVISTIC PICTURE

The classical Hamilton's equation of motion for the particle is given by [12]

$\dot{x} = [H, x]_{p,x}$ and

$\dot{p} = [H, p]_{p,x}$ (12)

where $[H, f]_{p,x}$ is the Poisson bracket and is defined by [12]

$[f, g]_{p,x} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}$ (13)

In this case f = x or p. In eqn.(12) the dots indicate the time derivative. Now using the relativistic version of Rindler Hamiltonian from eqn.(11), the explicit form of the equations of motion are given by

$\dot{x} = \left(1 + \frac{\alpha x}{c^2}\right)\frac{pc^2}{(p^2c^2 + m_0^2c^4)^{1/2}}$ and

$\dot{p} = -\frac{\alpha}{c}(p^2c^2 + m_0^2c^4)^{1/2}$ (14)

The parametric form of expressions for x and p represent the time evolution of spatial coordinate and the corresponding canonical momentum. The analytical expressions for time evolution of both the quantities can be obtained after integrating these coupled equations and are given by

$x = \frac{c^2}{\alpha}[C_0 \cosh(\omega t - \Phi) - 1]$ and

$p = -m_0 c \sinh(\omega t - \Phi)$ (15)

where C_0 and φ are the integration constants, which are real in nature and ω = α/c is the frequency defined for some kind of quanta in [13]. Hence eliminating the time coordinate, we can write

$\left(1 + \frac{\alpha x}{c^2}\right)^2 \frac{1}{C_0^2} - \frac{p^2}{m_0^2c^2} = 1$ (16)

This is the mathematical form of the set of classical trajectories of the particle in the phase space. Or in other words, these set of hyperbolas are the classical trajectories of the particle in the Rindler space. This is consistent with the hyperbolic motion of the particle in a uniformly accelerated frame. These set of hyperbolic equations can also be written as

$$p^2 = m_0^2 c^2 \left(\frac{2\alpha x}{c^2}\right)\left(1+\frac{x\omega}{2c}\right) \quad (17)$$

It is quite obvious from the parametric form of the variation of x and p with time that both the quantities are unbound. This is also reflected from the nature of phase space trajectories as shown in fig.(1) for the scaled x and p. The scaling factors are α/c^2 for x and $(m_0c)^{-1}$ for p. For the sake of illustration, we have chosen the arbitrary constant C0 = 1. In this figure we have also taken both the scaling factors identically equal to unity. Then obviously eqn.(16) reduces to

$$(x+1)^2 - p^2 = 1$$

We shall get the other set of trajectories by choosing different values for the scaling factors. It is obvious that in this case the centre of the hyperbola is at (−1,0). Therefore with the increase of α, the centre → (0,0). Further the vertices for this particular hyperbolic curve are at (0,0) and (−2,0). The second one is in scaled form. Therefore for the gravitational field α large enough, both the vertices coincide at the centre (0,0). Hence for very large values of α, these two curves touch each other at (0,0). We have therefore noticed that the phase space trajectories are unbound and consistent with the motion of the particle in Rindler space. Now in the Rindler coordinate system, the portion x >| t | of the Minkowski space is called the right Rindler wedge. The left Rindler wedge x <− | t | can be obtained by reflection. The null rays act as the event horizons for Rindler observers. An observer in the right wedge can not see any events in the left wedge. These two regions are causally disjoint two universes. However, exactly like the Minkowski space the past and the future can be defined and are causally connected.

B. NON-RELATIVISTIC SCENARIO

Let us now consider the non-relativistic form of Rindler Hamiltonian given by eqn.(11a). Now following eqn.(12), the equations of motion for the particle in Rindler space in the non-relativistic approximation are given by

$$\dot{x} = \left(1+\frac{\alpha x}{c^2}\right)\frac{p}{m_0} \text{and}$$

$$\dot{p} = -\frac{\alpha}{c^2}\left(\frac{p^2}{2m_0} + m_0c^2\right) \quad (18)$$

On integrating the second one we have

$$p = i2^{1/2} m_0 c \cot\left(\frac{2^{1/2}\omega t + \Phi}{2}\right) = ip_I \quad (19)$$

The particle momentum is therefore purely imaginary in nature with its real part $p_R = 0$. Here φ is a real constant phase. Next evaluating the first integral analytically, we have

$$x = \frac{c}{\omega}\left[-1+\cos\left\{ln\left(sin^2\left(\frac{2^{1/2}\omega t - \Phi}{2}\right)\right)\right\}\right] + i\frac{c}{\omega}\left[\sin\left\{ln\left(sin^2\left(\frac{2^{1/2}\omega t - \Phi}{2}\right)\right)\right\}\right]$$

$$x = x_R + ix_I \quad (20)$$

The spatial part is therefore complex in nature, where the real part

$$x_R = \frac{c}{\omega}\left[-1+\cos\left\{ln\left(sin^2\left(\frac{2^{1/2}\omega t - \Phi}{2}\right)\right)\right\}\right] \quad (21)$$

And the corresponding imaginary part is given by

$$x_I = \frac{c}{\omega}\left[\sin\left\{ln\left(sin^2\left(\frac{2^{1/2}\omega t - \Phi}{2}\right)\right)\right\}\right] \quad (22)$$

As before eliminating the time part, we have the mathematical form of phase space trajectories for the imaginary part only

$$p_I = 2^{1/2} m_0 c \frac{\left[1-exp\left\{\sin^{-1}\left(\frac{\omega}{c}x_I\right)\right\}\right]^{1/2}}{exp\left\{\frac{1}{2}\sin^{-1}\left(\frac{\omega}{c}x_I\right)\right\}} \quad (23)$$

Which gives the phase space trajectories of the particle in the Rindler space in non-relativistic scenario. It should be noted here that since the real part of the particle momentum is zero, we have considered the imaginary parts only. Since $-1 \le \sin^{-1}(a) \le +1$, $|\frac{\omega x_I}{c}| \le 1$, i.e, all possible values for x_I are not allowed.

In fig.(2) we have plotted the scaled x_R, which is $\left(\frac{\omega x_R}{c}\right)$ with scaled time $\left(\frac{\omega t}{2^{1/2}}\right)$ for $\Phi = 0$. Since the constant phase φ is completely arbitrary, for the sake of illustration we have chosen it to be zero. In this diagram the scaling factors are also chosen to be unity. Now if we consider variation of the scaling factors, the qualitative nature of the graphs will not change but there will be quantitative changes. In fig.(3) we have plotted the scaledx_I, i.e., $\left(\frac{\omega x_I}{c}\right)$ with scaled time $\left(\frac{\omega t}{2^{1/2}}\right)$ for $\Phi = 0$. In this case also the scaling factors are exactly equal to one. Further the same kind of variation as mentioned above will be observed for p_I with the change of scaling parameters. In fig.(5) the phase space trajectory for scaled x_I and scaled p_I is shown. Now$\sin^{-1}(a)$ can have values between −1 to +1 and further the quantity within the third bracket in the numerator must be ≥ 0 to make p_I real. Therefore the physically acceptable domain for scaled xI is from −1 to 0, we have shown in figs.(6) and (7) the plot of scaled x_I and scaled p_I with scaled time.

IV. BOHR-SOMMERFELD QUANTIZATION OF THE PARTICLE MOTION IN RINDLER SPACE

The condition for Bohr-Sommerfeld quantization is given by

$\oint pdx = n\hbar$ (24)

where n is the Bohr-Sommerfeld quantum number. In the present scenario, using the change of variable $X = 1 + \alpha x$, this can be written as

$\int_1^{X_m} pdX = \pi\alpha n\hbar$ (25)

To get it explicitly, we have replaced the particle momentum by its spatial coordinate using the expression for Rindler Hamiltonian. Then we have

$p = \left(\frac{E^2}{X^2} - m_0^2\right)^{1/2}$ Hence

$$E\oint\left(\frac{1}{X^2} - \frac{m_0^2}{E^2}\right)^{1/2} = \pi\alpha n\hbar$$

On substituting $X = {E\,cos\theta}/{m_0}$, with $X_m = E/m_0$, we have

$$-E\int_{X_m}^{0} \sin\theta \tan\theta \, d\theta$$

Hence

$E\left[lnE + ln\left\{1 + \left(1 - \frac{1}{E^2}\right)^{1/2}\right\} - \left(1 - \frac{1}{E^2}\right)^{1/2}\right] = \xi n$(26)

where E is redefined as E/m_0 and $\xi = \pi\alpha\hbar/m_0$. Solving numerically we have obtained E for various values of ξ. Using the χ^2minimization technique, have parametrized the energy eigen value in terms of the quantum number n, given by E = a+bn+cn^2, where a, b and c ξ dependent parameters. In the following we have given the tabular form of energy eigen values for various ξ.

ξ	a	b	c
0.01	1.09	0.02	-4*10^{-5}
1.0	1.93	0.48	-0.01
5.0	3.62	1.7	-0.02
50.0	12.98	10.5	-0.07

It is to be noted that such quadratic parametric form of energy eigen values are satisfied by the wave packets, which in the present picture oscillates between two classical turning points. So it is bounded and satisfy the necessary condition for Bohr-Sommerfeld quantization condition.

V. HEISENBERG EQUATION OF MOTION IN RINDLER SPACE

In this section we have developed a formalism for the Heisenberg equation of motion in Rindler space. Since the Hamiltonian is non-hermitian, i.e., $H = Xf(p) \neq H^{\dagger}$, we write

$$H(X,p) = \frac{1}{2}(Xf(p) + f(p)X) + \frac{1}{2}(Xf(p) - f(p)X)$$

$H(X,p) = H_h + H_{ah}$ (27)

where the symbols h and ah are for hermitian and antihermitian components. Then the time evolution operator

$U = exp(-iHt)$ (28)

is not unitary. The type of transformations with U is therefore not of unitary type. They are called the similarity transformations [14]. Although the Hamiltonian is not hermitian, it is PT symmetric, i.e., [H,PT] = 0. As a consequence, the energy eigen values are real [15] (see also [16]). The energy eigen spectrum for the Schrodinger equation has been observed to be real [17]. This is found to be solely because of the fact that H is PT-invariant. Now it is well know that$PxP^{-1} = -x, PpP^{-1} = -p$, whereas, $TpT^{-1} = -p, P\alpha P^{-1} = -\alpha$ and $T\alpha T^{-1} = \alpha$. Therefore it is a matter of simple algebra to show that $PT\,H\,(PT)^{-1} = H^{PT} = H$. As has been shown by several authors [15] that if H is PT-invariant, then the energy eigen values will be real. Here P and T are respectively the parity and the time reversal operators. Further if the Hamiltonian is PT symmetric, then H and PT should have common eigen states. In [13] we have noticed that the solution of the Schrodinger equation is obtained in terms of the variable$u = 1 + \alpha x/c^2$, which is PT-symmetric. Hence any function, e.g., Whittaker function $M_{k,\mu}(u)$ or Associated Laguerre function Ln m(u), the solution of the Schrodinger equation are PT-symmetric. These polynomials are also the eigen functions of the operator PT with eigen value +1. Of course with the replacement of hermiticity of the Hamiltonian with the PT-symmetry, we have not discarded the important quantum mechanical key features of the system described by this Hamiltonian and also kept the canonical quantization rule invariant, i.e., $TiT^{-1} = -i$. It should further be noted, which is the most important one, that under PT operation $\alpha \rightarrow -\alpha$. The whole problem will be shifted from right Rindler wedge to the left Rindler wedge, whereas the entire physics of the problem remain invariant. This is specially true in the present situation. Normally the left and the right Rindler wedges are not causally connected. The path connecting these two wedges is forbidden. Since the physics does not change, the energy eigen values or the energy spectra remain same after PT operation. In other words, this is happening because H is PT invariant. From the well known expression for Heisenberg equation of motion we have

$\dot{p} = -\alpha(p^2 + m_0^2)^{1/2}$ and

$$\dot{x} = \alpha x \tanh(\alpha t) + (1 - i)\left[\tanh(\alpha t) - \frac{\alpha}{2m_0}\frac{1}{cosh^3(\alpha t)}\right]$$

Integrating the differential equations involving time derivative of the momentum, the particle momentum, which is real in nature is given by

$$p = m_0 \sinh(\alpha t) \quad (29)$$

Whereas the spatial part can be obtained after integrating over time using the integrating factor $(\cosh(\alpha t))^{-1}$. Which is complex in nature and is given by

$$x = (1 - i)\left[\frac{1}{\alpha} - \frac{p}{2m_0^2}\right] = x_R + ix_I \quad (30)$$

where R and Iare respectively the real and imaginary parts of positional coordinate x. The real and the imaginary parts of the spatial coordinate are given by

$$x_R = \frac{1}{\alpha} - \frac{p}{2m_0^2} \text{and} x_I = -x_R \quad (31)$$

respectively. It can very easily be shown that for p → 0, ˙ p = $-m_0\alpha$, which is the Newton's second law of motion.

VI. ABSTRACT ALGEBRAIC METHOD IN RINDLER SPACE

In this study, we have considered only the nonrelativistic motion of the particle. In the non-relativistic approximation, the Rindler Hamiltonian is given by

$$H = (1 + \alpha x)\left(\frac{p^2}{2m_0} + m_0\right) \quad (32)$$

Let us now define [18

$$x = \beta(a^\dagger + a) \text{and} p = i\Upsilon(a^\dagger - a) (33)$$

where a and a† are the annihilation and creation operators respectively. We know that in the case of harmonic oscillator, $\beta = 1/(2m_0\omega_0)^{1/2}$ and $\Upsilon = ((m_0\omega_0/2))^{1/2}$, where ω_0 is the characteristic frequency. Here for the sake of simplicity, we use harmonic oscillator values for β and γ. Substituting x and p in the expression for nonrelativistic form of Hamiltonian as given above, we have

$$H = \frac{-1}{2m_0}\Upsilon^2\left[a^2 + a^{\dagger 2} + (aa^\dagger + a^\dagger a)\right] + m_0 + \alpha\beta\frac{1}{2m_0}\Upsilon^2(a + a^\dagger)\left[\left(a^2 + a^{\dagger 2}\right) + (aa^\dagger + a^\dagger a)\right] + \alpha\beta(a + a^\dagger)m_0 \quad (34)$$

Here the frequency is related to transition from some excited level to low lying states and is given by $\omega_0 = \alpha/c$. Since the only interaction here is the gravitational field,we actually what kind of radiation will emitted or absorbed by this type of transition, if any. Unlike the other conventional cases, here the frequency ω_0 is a constant and depends only on the magnitude of the uniform acceleration of the frame. Therefore this condition may act as some kind of selection rule for transition. In the above expression for the Hamiltonian, the part which is independent of α, the uniform acceleration, is H_0, the non-interacting part of the Hamiltonian, whereas the part depending on α is indicated by H′, the corresponding interaction part. In this case the only interaction is with the background constant gravitational field. It should be noted that the macroscopic gravity part goes into the quantum mechanical part through this interaction term and it becomes zero for α = 0. To obtain the total energy eigen value of the particle we treat the interaction term perturbatively. To be more specific, we consider time independent first order perturbation theory. We represent the nth state of the quantum mechanical system by $|n>$ and consider the commutation relation [a,a†] = 1. Then we have

$$E_n = < n|H_0|n > = < n\left|\left(a^\dagger a + \frac{1}{2}\right)\right|n > \omega_0 + m_0$$

$$E_n = \left(n + \frac{1}{2}\right)\omega_0 + m_0 \quad (35)$$

where n is the quantum number associated with nth state of the particle. Considering first order time independent perturbation theory, the total energy of the system is given by

$$W_n = E_n + H'_{nn} + \sum_m \frac{|H'|^2}{(E_n - E_m)} = E_n + \sum_m E'_{nm} \quad (36)$$

It can very easily be shown that the second term on the right hand side does not contribute in the total energy of the system. Only four terms contribute in the sum: For m = n − 1, the sum is

$$E'_n = \frac{\left[\alpha\beta n^{1/2}(m_0+\omega_0(n+1))\right]^2}{\omega_0} = E_n^1 \quad (37)$$

For m = n + 1, it is given by

$$E'_n = -\frac{\left[\alpha\beta n^{1/2}(m_0+n\omega_0)\right]^2}{\omega_0} = E_n^2 \quad (38)$$

For m = n − 2, we have

$$E'_n = \frac{\omega_0}{8}n(n-1)(\alpha\beta)^2 = E_n^3 \quad (39)$$

Whereas for m = n + 2, it is given by

$E_n^{'} = -\frac{\omega_0}{8} n(n+1)(\alpha\beta)^2 = E_n^4$ (40)

Now assuming that α is small enough (which may not be correct), the sum over m or in other words the perturbative series will be converging in nature. Therefore in principle it can be evaluated for all orders.

VII. MASSLESS PARTICLE IN RINDLER SPACE

Defining X = 1+αx, we have the expression for Hamiltonian for the massless particles

$H = Xp$ (41)

Although the massless particles do not exist in classical mechanics, for the sake of completeness we can obtain from the Hamilton's principle, the classical equation motion, given by

$\frac{dX}{dt} = X \text{and} \frac{dp}{dt} = -\alpha p$ (42)

The solutions are

$x = \frac{C_1 exp(\alpha t) - 1}{\alpha} \text{and} p = C_2 exp(-\alpha t)$ (43)

where C_1 and C_2 are integration constants. Since two Rindler wedges are not causally connected, $C_1 exp(\alpha t) - 1$ must be greater than zero. This will put some restriction on the absolute value of the constant C_1. In the quantum case, the eigen value equation is given by

$X\hat{p}|\psi> = E|\psi>$ (44)

Hence we can write

$-i\alpha X = \frac{d\psi(x)}{dX} = E\psi(x)$ (45)

where X is the eigen value of the operator $\hat{X}$. The solution of the above equation, which indicates the eigen state of the particle is given by

$\psi(x) = \psi_0 exp\left(\frac{iE}{\alpha} lnX\right)$ (46)

Which can also be written as

$\psi(x) = \psi_0 X^{iE/\alpha}$ (47)

Again assuming that the acceleration α of the frame is small enough, we can re-write the wave function in the form

$\psi(x) = \psi_0 exp(i\alpha x)$ (48)

Here ψ_0 is the box normalization constant in one dimension.

VIII. CONCLUSION

In this article we have developed basic quantum mechanics in Rindler space. The Rindler Hamiltonian is non-hermitian but is found to be PT-symmetric. We have noticed that the energy eigen values are real in nature in all these studies.

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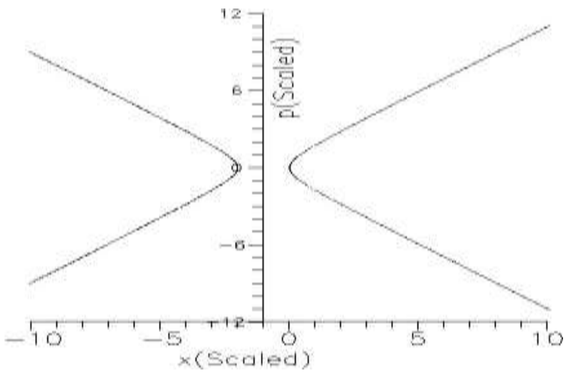

FIG. 1: Phase space trajectories for the relativistic scenario with the scaling parameters equal to unity

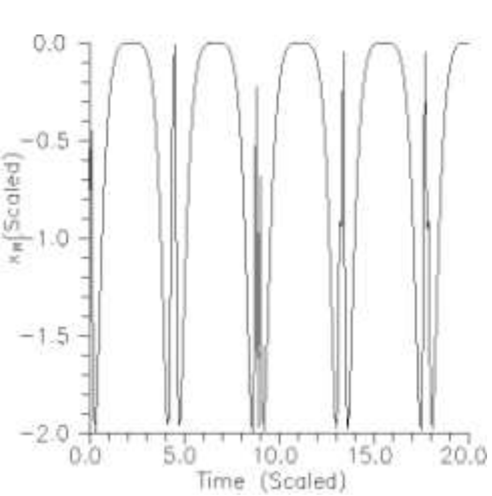

FIG. 2: Variation of scaled x_R with scaled time

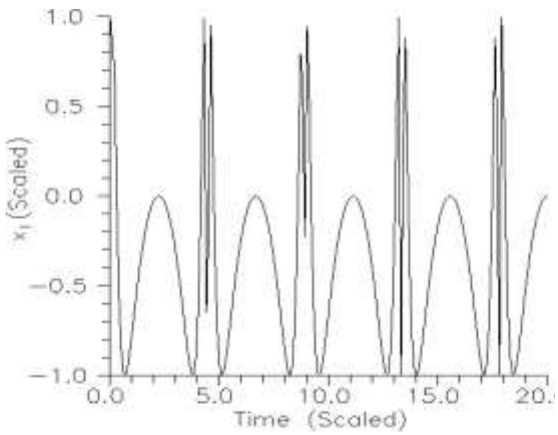

FIG. 3: Variation of scaled x_I with scaled time

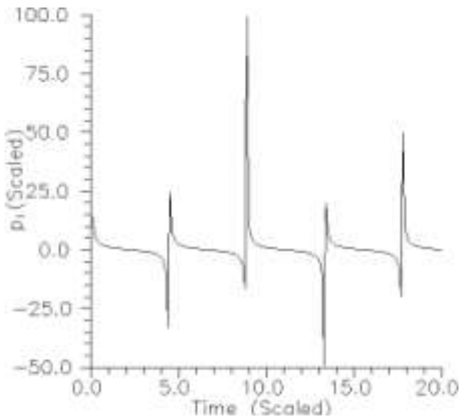

FIG. 4: Variation of scaled p_I with scaled

time

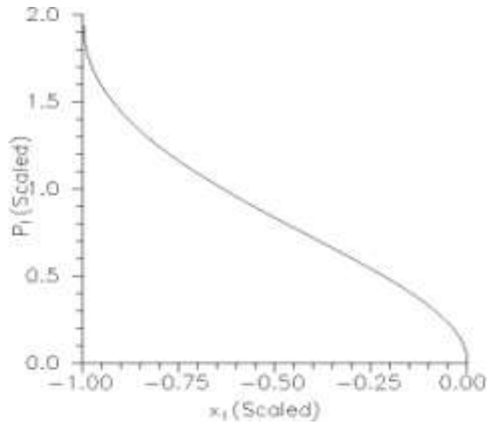

FIG. 5 Phase space trajectories for the non-relativistic scenario with the scaling parameters equal to unity.

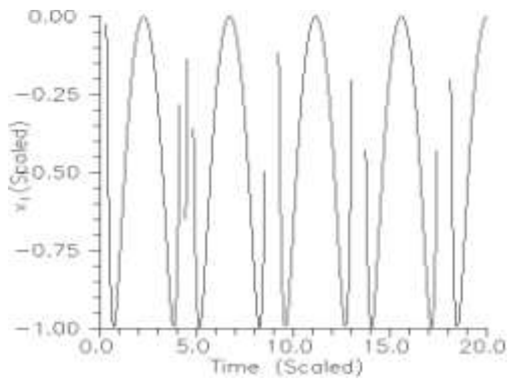

FIG. 6: Temporal variation of x_I in physically acceptable domain

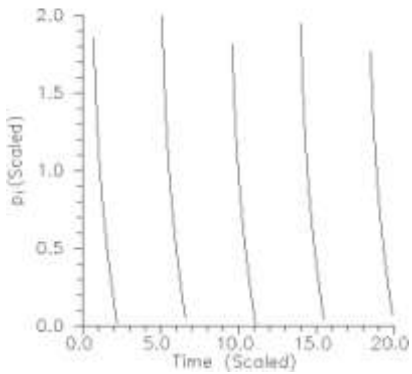

FIG. 7: Temporal variation of p_I in physically acceptable domain