An Extended Study On The Precession Equation In The Gravitational Field of Spherical Mass

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Abstract

In this research work, an extended Newton's planetary equation of motion is solved analyticallyto obtain the precession equation, orbital eccentricity and amplitude in the gravitational field of spherical bodies.The results obtained showed that the precession equation reduces to exactly Einstein's precession equation. The orbital eccentricity of the orbit of the planets and amplitude reduces to c^0 *to the corresponding pure Newtonian.It contains post Newtonian and Einstein correction terms to all orders of*c^{-2}. *Theseresults point to the fact that the extended Newton's dynamical field equation can be used to effectively derivedthe precession equation in the gravitational field of spherical bodies.*

Keywords: *Newton's, Planetary Equation, Precession, Eccentricity, Amplitude, Gravitational Field, Spherical.*

I. Introduction

In 1686 Newton published his dynamical theory of gravitation. This theory was successful in explaining the gravitational phenomena on earth and some of the experimental facts of the solar system [1, 2]. At the end of the 19th century there were several attempts to extend or generalized Newton's dynamical theory of gravitation in order to provide better agreement to all physical theories [1 - 4].

In 1915 Einstein published his geometrical theory of gravitation which is popularly known as General Relativity (GR). This theory offered a resolution of the anomalous orbital precession of the orbit of the planet as well as the gravitational spectral shift by the sun [1-4].

Despite the famous test of General Relativity (GR) all the authorities in Relativity and Physics in general have continued to raise objections against the mathematical difficulty of Einstein's theory of Relativity. A number of Physicists still hold on the view that Newton's dynamical theory of gravitation can be extended in such a way as to account satisfactory for the experimental data and phenomena [1, 2]

On the basis of this, we have in our previous papers [5, 6], used the Riemannian geometry togeneralized or extend Newton's dynamical field equation to obtain generalized dynamical

(i) Field equation and gravitational scalar potential exterior and interior for static homogeneous spherical bodies.
(ii) Planetary equation of motion for static homogeneous spherical bodies.

In this paper the extended Newton's planetary equation of motion for static homogeneous spherical bodies were used to derive the precession equation in the gravitational field of static homogeneous spherical bodies.

II. Methodology

The well - known Newton's and Einstein's planetary equation of motion for static homogeneous spherical bodies is given explicitly as [1 - 3]

$$\frac{d^2u}{d\phi^2}+u=\frac{k}{l^2} \qquad (1)$$

and

$$\frac{d^2u}{d\phi^2}+u-\frac{3ku^2}{c^2}=\frac{k}{l^2} \qquad (2)$$

An extended Newton's planetary equation of motion in the gravitational field of spherical bodies is shown to be given explicitly as [5, 6]

$$\frac{d^2u}{d\phi^2} + \left(1 + \frac{6k^2}{c^2l^2}\right)u = \frac{\kappa}{l^2}\left(1 - \frac{3k}{5c^2R}\right) \quad (3)$$

where, ϕ and u are the instantaneous angular and reciprocal radial displacement of the planet in the fixed plane of motion, with the sun as origin, l is the constant angular momentum per unit mass and c is the speed of light in vacuum.

and

$$k = GM$$

where, M is the rest mass of the sun, G is the universal gravitational constant.

Theextended Newton's planetary equation of motion is solved analytically to obtain the precession equation and hence the orbital eccentricity and amplitude in the gravitational field of spherical bodies.

III. Results and Discussions

Supposing that the analytical solution of equation (3)is the form of Taylor series given as

$$u(\phi) = \sum_{n=1}^{\infty} A_n exp\{ni(\omega\phi + \alpha)\} \quad (4)$$

where, A_n, ω and α are arbitrary constants.

Equation (4) can be expanded to give

$$u(\phi) = \sum_{n=1}^{\infty} A_n exp\{ni(\omega\phi + \alpha)\}$$

$$= A_0 + A_1 exp\{i(\omega\phi + \alpha)\}$$

$$+A_2 exp\{2i(\omega\phi + \alpha)\}$$

$$+A_3 exp\{3i(\omega\phi + \alpha)\} + \cdots \quad (5)$$

Differentiating equation (5) twice with respect to $\phi$and substituting into equation (3) gives

$$-\omega^2 A_1 exp\{i(\omega\phi + \alpha)\} - 4\omega^2 A_2 exp\{2i(\omega\phi + \alpha)\} + 9\omega^2 A_3 exp\{3i(\omega\phi + \alpha)\}$$

$$+ \left(1 + \frac{6k^2}{c^2l^2}\right)[A_0 + A_1 exp\{i(\omega\phi + \alpha)\} + A_2 exp\{2i(\omega\phi + \alpha)\} + \cdots]$$

$$= \frac{k}{l^2}\left(1 - \frac{3k}{5c^2R}\right) \quad (6)$$

Equating constant terms of equation (6), we obtain

$$\left(1 + \frac{6k^2}{c^2l^2}\right)A_0 = \frac{k}{l^2}\left(1 - \frac{3k}{5c^2R}\right)$$

Making A_0 the subject of the formula yields

$$A_0 = \frac{k}{l^2}\left(1 - \frac{3k}{5c^2R}\right)\left(1 + \frac{6k^2}{c^2l^2}\right)^{-1} \quad (7)$$

Equating the coefficient of first order exponential terms of equation (6) we get

$$\omega^2 = \left(1 + \frac{6k^2}{c^2l^2}\right)$$

By squaring both sides, we obtain

$$\omega = \pm\left(1 + \frac{6k^2}{c^2l^2}\right)^{\frac{1}{2}} \quad (8)$$

Equation (8) will give two solutions, which is given explicitly as

$$\omega_\alpha = \left(1 + \frac{6k^2}{c^2l^2}\right)^{\frac{1}{2}} \quad (9)$$

and

$$\omega_\beta = -\left(1 + \frac{6k^2}{c^2l^2}\right)^{\frac{1}{2}} \quad (10)$$

Equation (10) is mathematicallysound but physically of no significance as it yields a complex solution. Hence, we consider equation (10) as our physical expression for angular velocity.

Substituting equation (7) into equation (5), we obtain

$$u(\phi) = \frac{k}{l^2}\left(1 - \frac{3k}{5c^2R}\right)\left(1 + \frac{6k^2}{c^2l^2}\right)^{-1}$$

$$+A_1 exp\{i(\omega\phi + \alpha)\} + \cdots \qquad (11)$$

where,ω is the orbital angular frequency.

In this case, the exact analytical solution of generalized Newton's planetary equation in a complex function of ϕ may be written in Cartesian form as [7]

$$u(\phi) = x(\phi) + iy(\phi)(12)$$

where

$$x(\phi) = A_0 + A_1\cos(\omega_1\phi + \phi_0)$$

$$+f_2(A_1)\cos 2[(\omega_1\phi + \phi_0)] + \cdots(13)$$

and

$$y(\phi) = A_0 + A_1\sin(\omega_1\phi + \phi_0)$$

$$+f_2(A_1)\sin 2[(\omega_1\phi + \phi_0)] + \cdots(14)$$

Therefore it may be expressed in Euler form as

$$u(\phi) = R(\phi)e^{i\Phi(\phi)} \qquad (15)$$

where,R is the magnitude given by

$$R(\phi) = [x^2(\phi) + y^2(\phi)]^{\frac{1}{2}} \qquad (16)$$

andΦ is the argument given by

$$\Phi(\phi) = \tan^{-1}\left[\frac{y(\phi)}{x(\phi)}\right] \qquad (17)$$

Hence by definition of instantaneous radial coordinate of the planet from the sun, r is given by

$$r(\phi) = R^{-1}(\phi)e^{-i\Phi(\phi)} \qquad (18)$$

Therefore the magnitude of the instantaneous complex radial displacement of the planet from the sun can be considered to be real physically measurable instantaneous radial displacement,r_p. Thus

$$r_p(\phi) = R^{-1}(\phi) = [x^2(\phi) + y^2(\phi)]^{-\frac{1}{2}} \qquad (19)$$

Neglecting the terms in $f_n(A_1)$for $n > 1$. Then it follows from (14), (15) and (19) that

$$r_p(\phi) = \frac{A}{1+\varepsilon_1\cos(\omega_1\phi+\phi_0)} \qquad (20)$$

$$A = \frac{1}{A_{0-}}\left(1 + \frac{A_1^2}{A_{0-}^2}\right) \quad (21)$$

and

$$A = \frac{A_1}{A_{0-}}\left(1 + \frac{A_1^2}{A_{0-}^2}\right)^{-1} (22)$$

Consequently, the orbit is a precesing conic section with eccentricity and hence semi-major axis given by

$$A = \left(\frac{A}{1-\varepsilon_1^2}\right) \qquad (23)$$

The perihelion displacement angle Δ is known to be given by [1, 2, 8]

$$\Delta = 2\pi\left(\omega_\beta^{-1} - 1\right) \qquad (24)$$

Substituting equation (10) into (24) yield

$$\Delta = 2\pi\left[-\left(1 + \frac{6k^2}{c^2l^2}\right)^{-1}\right]^{\frac{1}{2}} - 2\pi(25)$$

Or

$$\Delta = 2\pi\left[-\left(1+\frac{6k^2}{c^2l^2}\right)^{-\frac{1}{2}}\right] - 2\pi \quad (26)$$

By using binomial expansion and simplifying, we obtain

$$\Delta = \frac{6k^2\pi}{c^2l^2} - 4\pi \quad (27)$$

Using the first order approximation method equation (27) reduces to

$$\Delta = \frac{6k^2\pi}{c^2l^2} \qquad (28)$$

It follows from equation (11) that the generalized Newton's orbital amplitude and frequency is given explicitly as

$$A = \frac{1}{r_i} - \frac{k}{l^2}\left(1-\frac{3k}{5c^2R}\right)\left(1+\frac{6k^2}{c^2l^2}\right)^{-1} \quad (29)$$

and

$$\varepsilon = 1 - \frac{l^2}{r_ik}\left(1-\frac{3k}{5c^2R}\right)^{-1}\left(1+\frac{6k^2}{c^2l^2}\right) \quad (30)$$

In the order of c^0 these results reduce to the corresponding pure Newtonian calculations [1, 2,9]

$$\frac{1}{r_i} - \frac{k}{l^2}, \quad 1 - \frac{l^2}{r_ik}$$

III. Conclusions

We have in this researchwork shown how to derive the precession equation, orbital amplitude and orbital frequency in the gravitational field of a spherical body using an extended Riemannian planetary equation of motion. The precession equation, orbital amplitude and frequency in the gravitational field of a spherical body are found to be equations (28), (29) and (30). It is interesting and instructive to note that the precession equation (28) reducesto exactly Einstein equation while the orbital amplitude (29) and frequency (30) reduces to c^0 to the corresponding pure Newtonian terms. It contains post Newton's and Einstein's correctionterms to all orders ofc^{-2}. Consequently, the additional correction terms to the orbital amplitude and frequency are open up for theoretical development and experimental investigations and applications. The resultsobtained in this research work point to the fact that it is possible to derive the precession equation by generalising or extending Newton's dynamical field equation.These results can be applied to studythe steady change in the orientation of the axis of a rotation of the earth and hence inthe study of weather conditions.

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