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Original Article

Electron Trajectories from One Bohr-Sommerfeld Orbit to Another One

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***Abstract** — In previous papers we have studied 3D two-body problem of classical electrodynamics based on the extended Synge's model (with a new form of the Dirac radiation term) and proved an existence-uniqueness of a periodic orbit . Later we have investigated the Kepler problem for of two charched particles using polar coordinates -in the plain of motion. In this way we have showed an existence of the Bohr-Sommerfeld stationary states. Here we show an existence of orbits of transition of the moving particle (electron) from one stationaty state to another one. This is made by a suitable choice of function space and applying the fixed ponit method.*

Keywords — *Bohr-Sommerfeld stationary state, fixed point method, Kepler problem, operator method for transition orbits, two-body problem of classical electro-dynamics.*

I. INTRODUCTION

The present paper is one of the series of articles devoted to the two-body problem of classical electrodynamics [1]-[5]. In [2] we have derived equations of motion describing two charged particles based on the retarded model of J. L. Synge [6] (cf. also [7]) and the extended Dirac radiation term [1]. In [3] we have proved an existence-uniqueness of a periodic solution of the two-body problem and in [4] we have extended and improved the results obtained in [3]. In [5] we have proved an existence-uniqueness of periodic solution of the Kepler problem in polar coordinates. In this way we have shown the existence-uniqueness of the Bohr-Sommerfeld stationary states in the plane (cf. [8]-[11]). In order to show that the periodic trajectories are not isolated from the general motion of the particles (electrons) orbiting the nucleus, we must study the trajectories of the electron from one stationary state to another one. This is the main goal of the present paper − to prove an existence of transition trajectories of a particle (for instance of electron in the Hydrogen atom) from the first Bohr orbit to the second one and so on. We remind that in quantum mechanics there are no trajectories. The electron, for instance, can be found on a given place of the space with prescribed probability.

Here we consider only the second group of equations from [2] which is substantiated in [5]. In [5] we proceed assigning to the Kepler problem an operator whose fixed points are periodic solutions in the space of velocities. We introduce a suitable function space and define an operator acting on this space. Its fixed point is a solution in the space of velosities. The corresponding trajectories are such that the particles move periodically and then they "jump" (but continuously) on another orbit with larger radius − namely the second Bohr orbit. These solutions correspond to the Bohr-Sommerfeld stationary states [8]-[11] which could be more general-not only in circles or ellipses.

We would like to note some papers concerning similar problems by different approaches [12]-[17].

The paper consists of six sections. In Section II we recall from [5] the derivation of the equations of motion in the plane Ox_2x_3 and present them in polar coordinates. In Section III we show that the suitable choice of the function spaces allows us to obtain orbits of transition from a stationary state with given radius to a stationary state with larger radius. In order to smooth out the jumps in the derivatives of polar radius we apply the technique from the theory of generalized functions [18]-[22]. In Section IV we define the operator corresponding to our problem and prove the Main Lemma- our problem (existence of transition orbits) has a solution iff the operator has a fixed point. In Section V we prove the main existence-uniqueness theorem, namely, if some inequalities are satisfied, there exists a unique trajectory that passes from one stationay state to another with a larger radius. The proof is based on the fixed point method [23]. In Section VI we give a numerical example for hydrogen atom (cf. [24]) which confirms the results obtained. Moreover, we transform the energy term from [25] and find an interval in which it can vary. Then we notice that the energies of all stationary states are included in it.

First of all, we recall the basic denotations and results from [2]. The system of equations of motion in the Minkowski space with radiation terms is

$$\begin{aligned} \frac{d\lambda_r^{(1)}}{ds_1} &= \frac{e_1}{m_1 c^2}\left(F_{rs}^{(2)}\lambda_s^{(1)} + F_{rs}^{(1)rad}\lambda_s^{(1)}\right) \\ \frac{d\lambda_r^{(2)}}{ds_2} &= \frac{e_2}{m_2 c^2}\left(F_{rs}^{(1)}\lambda_s^{(2)} + F_{rs}^{(2)rad}\lambda_s^{(2)}\right) \end{aligned} \qquad (1)$$

where e_1, e_2 are charges, m_1, m_2 – masses of the moving particles, c – the vacuum speed of light, $F_{rs}^{(p)}$ $(p=1,2)$ – the componenets of the electromagnetic tensor, $\frac{d\lambda_r^{(p)}}{ds_p}$ – the unit tangent vector to the world line. Following [2], [3] we present (1) in the form

$$\frac{du_1^{(1)}}{dt}=\frac{\Delta_1^2}{c^2}\frac{(c^2-u_1^{(1)}u_1^{(1)})G_1^{(12)}-u_1^{(1)}u_2^{(1)}G_2^{(12)}-u_1^{(1)}u_3^{(1)}G_3^{(12)}}{\Delta_1^2}+G_1^{(1)rad}$$

$$\frac{du_2^{(1)}}{dt}=\frac{\Delta_1^2}{c^2}\frac{-u_2^{(1)}u_1^{(1)}G_1^{(12)}+(c^2-u_2^{(1)}u_2^{(1)})G_2^{(12)}-u_2^{(1)}u_3^{(1)}G_3^{(12)}}{\Delta_1^2}+G_2^{(1)rad}$$

$$\frac{du_3^{(1)}}{dt}=\frac{\Delta_1^2}{c^2}\frac{-u_3^{(1)}u_1^{(1)}G_1^{(12)}-u_3^{(1)}u_2^{(1)}G_2^{(12)}+(c^2-u_3^{(1)}u_3^{(1)})G_3^{(12)}}{\Delta_1^2}+G_3^{(1)rad} \tag{2}$$

$$\frac{du_1^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{(c^2-u_1^{(2)}u_1^{(2)})G_1^{(21)}-u_1^{(2)}u_2^{(2)}G_2^{(21)}-u_1^{(2)}u_3^{(2)}G_3^{(21)}}{\Delta_p^2}+G_1^{(2)rad}$$

$$\frac{du_2^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{-u_2^{(2)}u_1^{(2)}G_1^{(21)}+(c^2-u_2^{(2)}u_2^{(2)})G_2^{(21)}-u_2^{(2)}u_3^{(2)}G_3^{(21}}{\Delta_2^2}+G_2^{(2)rad}$$

$$\frac{du_3^{(2)}}{dt}=\frac{\Delta_2^2}{c^2}\frac{-u_3^{(2)}u_1^{(2)}G_1^{(21)}-u_3^{(2)}u_2^{(2)}G_2^{(21)}+(c^2-u_3^{(2)}u_3^{(2)})G_3^{(21)}}{\Delta_2^2}+G_3^{(2)rad} \tag{3}$$

Here $\vec{x}^{(p)}(t)=\left(x_1^{(p)}(t),x_2^{(p)}(t),x_3^{(p)}(t)\right)$, $\vec{u}^{(p)}(t)=\left(u_1^{(p)}(t),u_2^{(p)}(t),u_3^{(p)}(t)\right)$ $(p=1,2)$ are unknown trajectories and velocities of the charged particles and $\left\langle \vec{a},\vec{b}\right\rangle=a_1b_1+a_2b_2+a_3b_3$ is the dot product in 3-dimensional Eucleadian subspace.

As we have already mentioned we consider only the second group of equations (3) with argument t and neglect the equations (2) with retarded arguments (cf. [5]). Since we consider the Kepler problem, the first particle is stationary one and it is stated at the origin $O(0,0,0)$, that is,

$$P_1=\left(x_1^{(1)}(t)=0, x_2^{(1)}(t)=0, x_3^{(1)}(t)=0\right),\ t\in(-\infty,\infty).$$

The Cartesian coordinates of the second particle are $P_2=\left(x_1^{(2)}(t)=0,\ x_2^{(2)}(t),\ x_3^{(2)}(t)\right)$, $t\in(-\infty,\infty)$ and its motion is in the plane Ox_2x_3. Obviously

$$u_1^{(1)}(t)=0,\ u_2^{(1)}(t)=0,\ u_3^{(1)}(t)=0;\ \dot{u}_1^{(1)}(t)=0,\ \dot{u}_2^{(1)}(t)=0,\ \dot{u}_3^{(1)}(t)=0.$$

We use the denotations from [2]:

$$\Delta_1=\sqrt{c^2-\left\langle \vec{u}^{(1)}(t),\vec{u}^{(1)}(t)\right\rangle};$$

$$\Delta_{21}=\sqrt{c^2-\left\langle \vec{u}^{(1)}(t-\tau_{21}),\vec{u}^{(1)}(t-\tau_{21})\right\rangle}=c;$$

$$\vec{\xi}^{(21)}=\left(\xi_1^{(21)},\xi_2^{(21)},\xi_3^{(21)}\right)=$$
$$=(x_1^{(2)}(t)-x_1^{(1)}(t-\tau_{21}),\ x_2^{(2)}(t)-x_2^{(1)}(t-\tau_{21}),\ x_3^{(2)}(t)-x_3^{(1)}(t-\tau_{21}))=$$
$$=(0,\ x_2^{(2)}(t),\ x_3^{(2)}(t))$$

where $\xi^{(21)}=\left(\xi_1^{(21)},\xi_2^{(21)},\xi_3^{(21)},ic\tau_{21}\right)$ is an isotropic vector, that is, $\left\langle \xi^{(21)},\xi^{(21)}\right\rangle_4=0$ (the dot product in the Minkowski space) which yields a functional equation $c\tau_{21}=\sqrt{\left\langle \vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}$ for τ_{21}. Further on, we have (cf. [2]):

$$\Delta_2=\sqrt{c^2-\left\langle \vec{u}^{(2)}(t),\vec{u}^{(2)}(t)\right\rangle};$$

$$D_{21}=\frac{c\sqrt{\left\langle \vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}-\left\langle \vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle}{c\sqrt{\left\langle \vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}-\left\langle \vec{\xi}^{(21)},\vec{u}^{(2)}\right\rangle}=\frac{c^2\tau_{21}}{c^2\tau_{21}-\left\langle \vec{\xi}^{(21)},\vec{u}^{(2)}\right\rangle};$$

$$M_{21}=1+D_{21}\frac{\Delta_{21}^2\left\langle \vec{\xi}^{(21)},\dot{\vec{u}}^{(1)}\right\rangle+\left(\left\langle \vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21}\right)\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle}{\Delta_{21}^4}=1;$$

$$G_\alpha^{(21)}=\frac{e_1e_2\Delta_2}{m_2c^2}\left[M_{21}\Delta^2{}_{21}\frac{(\left\langle \vec{u}^{(2)},\vec{u}^{(1)}\right\rangle-c^2)\,\xi_\alpha{}^{(21)}+(\left\langle \vec{\xi}^{(21)},\vec{u}^{(2}\right\rangle-c^2\tau_{21})u_\alpha{}^{(1)}}{(\left\langle \vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21})^3}-\right.$$

$$-\frac{D_{21}}{\Delta_{21}^2(\left\langle \vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21})^2}\left(\Delta_{21}^2\left\langle \vec{u}^{(2)},\dot{\vec{u}}^{(1)}\right\rangle\xi_\alpha{}^{(21)}+(\left\langle \vec{u}^{(2)},\vec{u}^{(1)}\right\rangle-c^2)\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\xi_\alpha{}^{(21)}-\right.$$

$$\left.\left.-(\left\langle \vec{\xi}^{(21)},\vec{u}^{(2)}\right\rangle-c^2\tau_{21})(\Delta_{21}^2\dot{u}_\alpha{}^{(1)}+u_\alpha{}^{(1)}\left\langle \vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle)\right)\right]=\frac{e_1e_2\Delta_2}{m_2}\frac{\xi_\alpha{}^{(21)}}{c^4\tau_{21}{}^3};$$

$$G_\alpha^{(2)rad}=-\frac{e_2^2}{m_2c^2\Delta_2}\left(\frac{u_\alpha^{(2)}(t)}{\Delta_2{}^2}\left\langle \vec{u}^{(2)}(t),\ddot{\vec{u}}^{(2)}(t)\right\rangle+\ddot{u}_\alpha^{(2)}(t)\right)$$

$\alpha=2,3$.

So we reach the following system on $[0,\infty)$:

$$\begin{aligned}\frac{du_2^{(2)}}{dt}&=\frac{e_1e_2\Delta_2}{m_2}\left(\frac{c^2-u_2^{(2)}u_2^{(2)}}{c^2}\frac{\xi_2^{(21)}}{c^4\tau_{21}{}^3}-\frac{u_2^{(2)}u_3^{(2)}}{c^2}\frac{\xi_3^{(21)}}{c^4\tau_{21}{}^3}\right)-\\&-\frac{e_2^2}{m_2c^2\Delta_2}\left(\frac{u_2^{(2)}(t)}{\Delta_2{}^2}\left\langle \vec{u}^{(2)}(t),\ddot{\vec{u}}^{(2)}(t)\right\rangle+\ddot{u}_2^{(2)}(t)\right)\\\frac{du_3^{(2)}}{dt}&=\frac{e_1e_2\Delta_2}{m_2}\left(\frac{-u_3^{(2)}u_2^{(2)}}{c^2}\frac{\xi_2^{(21)}}{c^4\tau_{21}{}^3}+\frac{c^2-u_3^{(2)}u_3^{(2)}}{c^2}\frac{\xi_3^{(21)}}{c^4\tau_{21}{}^3}\right)-\\&-\frac{e_2^2}{m_2c^2\Delta_2}\left(\frac{u_3^{(2)}(t)}{\Delta_2{}^2}\left\langle \vec{u}^{(2)}(t),\ddot{\vec{u}}^{(2)}(t)\right\rangle+\ddot{u}_3^{(2)}(t)\right).\end{aligned} \tag{4}$$

II. EQUATIONS OF MOTION WITH RADIATION TERMS IN POLAR COORDINATES

Passing to the polar coordinates in Ox_2x_3 we obtain

$$P_2=\left(x_1^{(2)}(t);x_2^{(2)}(t);x_3^{(2)}(t)\right)=(0,\ \rho(t)\cos\varphi(t),\ \rho(t)\sin\varphi(t));$$

$$\vec{\xi}^{(21)}=(0,\ \rho(t)\cos\varphi(t),\ \rho(t)\sin\varphi(t));\ \tau_{21}=\rho(t)/c;$$

$$\Delta_2=\sqrt{c^2-\dot{\rho}^2-\rho^2\dot{\varphi}^2};\ \Delta_{21}=c;$$

$$\left(u_1^{(2)}(t), u_2^{(2)}(t), u_3^{(2)}(t)\right) = \left(0, \dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi\ ,\ \dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi\right)$$
$$\left(\dot u_1^{(2)}(t), \dot u_2^{(2)}(t), \dot u_3^{(2)}(t)\right) =$$
$$= \left(0, \left(\ddot\rho - \rho\dot\varphi^2\right)\cos\varphi - \left(2\dot\rho\dot\varphi + \rho\ddot\varphi\right)\sin\varphi,\ \left(\ddot\rho - \rho\dot\varphi^2\right)\sin\varphi + \left(2\dot\rho\dot\varphi + \rho\ddot\varphi\right)\cos\varphi\right)$$
$$\left(\ddot u_1^{(2)}(t)\ddot u_2^{(2)}(t), \ddot u_3^{(2)}(t)\right) =$$
$$= \left(0, \left(\dddot\rho - 3\dot\rho\dot\varphi^2 - 3\rho\dot\varphi\ddot\varphi\right)\cos\varphi - \left(3\ddot\rho\dot\varphi - \rho\dot\varphi^3 + 3\dot\rho\ddot\varphi + \rho\dddot\varphi\right)\sin\varphi\ ,\right.$$
$$\left.\left(3\ddot\rho\dot\varphi - \rho\dot\varphi^3 + 3\dot\rho\ddot\varphi + \rho\dddot\varphi\right)\cos\varphi + \left(\dddot\rho - 3\dot\rho\dot\varphi^2 - 3\rho\dot\varphi\ddot\varphi\right)\sin\varphi\right)$$

$$\left\langle \vec\xi^{(21)}, \vec u^{(2)}\right\rangle = \rho\dot\rho\ ;\ D_{21} = \frac{c}{c-\dot\rho}\ ;\ \left\langle \vec u^{(1)}, \vec u^{(2)}\right\rangle = 0;$$
$$\left\langle \vec u^{(2)}, \vec u^{(2)}\right\rangle = \dot\rho^2 + \rho^2\dot\varphi^2\ ;\left\langle \vec\xi^{(12)}, \dot{\vec u}^{(2)}\right\rangle = \rho^2\dot\varphi^2 - \rho\ddot\rho\ ;$$
$$\left\langle \vec u^{(2)}, \dot{\vec u}^{(2)}\right\rangle = \dot\rho\ddot\rho + \rho\dot\rho\dot\varphi^2 + \rho^2\dot\varphi\ddot\varphi\ ;$$
$$\left\langle \vec u^{(2)}(t), \ddot{\vec u}^{(2)}(t)\right\rangle = \dot\rho\dddot\rho + \rho^2\dot\varphi\dddot\varphi + 3\rho\ddot\rho\dot\varphi^2 - 3\dot\rho^2\dot\varphi^2 - \rho^2\dot\varphi^4$$

Here we present equations (4) in polar coordinates

$$\ddot\rho\cos\varphi - \rho\ddot\varphi\sin\varphi =$$
$$= \rho\dot\varphi^2\cos\varphi + 2\dot\rho\dot\varphi\sin\varphi + \frac{c^2 - \left(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi\right)^2}{c^2} G_2^{(21)} -$$
$$- \frac{\left(\dot\rho^2 - \rho^2\dot\varphi^2\right)\sin 2\varphi + 2\rho\dot\rho\dot\varphi\cos 2\varphi}{2c^2} G_3^{(21)} + G_2^{(2)rad}$$
$$\ddot\rho\sin\varphi + \rho\ddot\varphi\cos\varphi =$$
$$= \rho\dot\varphi^2\sin\varphi - 2\dot\rho\dot\varphi\cos\varphi - \frac{\left(\dot\rho^2 - \rho^2\dot\varphi^2\right)\sin 2\varphi + 2\rho\dot\rho\dot\varphi\cos 2\varphi}{2c^2} G_2^{(21)} +$$
$$+ \frac{c^2 - \left(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi\right)^2}{c^2} G_3^{(21)} + G_3^{(2)rad}.$$

Under assumption $\rho(t) \ge \rho_0 > 0$ the last system can be solved with respet to $\ddot\rho, \ddot\varphi$.

In view of $G_2^{(21)} = \frac{e_1e_2}{m_2c}\ \frac{\sqrt{c^2 - \dot\rho^2 - \rho^2\dot\varphi^2}}{\rho^2}\cos\varphi$;

$G_3^{(21)} = \frac{e_1e_2}{m_2c}\frac{\sqrt{c^2 - \dot\rho^2 - \rho^2\dot\varphi^2}}{\rho^2}\sin\varphi$ we obtain

$$\ddot\rho = \rho\dot\varphi^2 + \frac{e_1e_2}{m_2c}\ \frac{\sqrt{c^2 - \dot\rho^2 - \rho^2\dot\varphi^2}}{\rho^2}\ \frac{c^2 - \dot\rho^2}{c^2} +$$
$$+ G_2^{(2)rad}\cos\varphi + G_3^{(2)rad}\sin\varphi;$$
$$\ddot\varphi = -\frac{2\dot\rho\dot\varphi}{\rho} - \frac{e_1e_2}{m_2c}\ \frac{\dot\rho\dot\varphi\sqrt{c^2 - \dot\rho^2 - \rho^2\dot\varphi^2}}{\rho^2}\ \frac{c^2 - \dot\rho^2}{c} +$$
$$+ G_3^{(2)rad}\cos\varphi - G_2^{(2)rad}\sin\varphi\ , t \ge 0.$$

We recall the derivation of the radiation terms (cf. [5]):

$$\rho(t+\tau) \approx \rho(t)\ ;\ \varphi(t+\tau) \approx \varphi(t)\ ;$$
$$\dot\rho(t+\tau) \approx \dot\rho(t)\ ; \dot\varphi(t+\tau) \approx \dot\varphi(t)\ ;$$

$$\frac{\dot u_2^{(2)}(t+\tau) - \dot u_2^{(2)}(t-\tau)}{2\tau} \approx$$
$$\approx \frac{\ddot\rho(t+\tau) - \ddot\rho(t-\tau)}{2\tau}\cos\varphi(t) - \frac{\ddot\varphi(t+\tau) - \ddot\varphi(t-\tau)}{2\tau}\rho(t)\sin\varphi(t) \approx$$
$$\approx \dddot\rho\cos\varphi - \dddot\varphi\rho\sin\varphi;$$

$$\frac{\dot u_3^{(2)}(t+\tau) - \dot u_3^{(2)}(t-\tau)}{2\tau} \approx$$
$$\frac{\ddot\rho(t+\tau) - \ddot\rho(t-\tau)}{2\tau}\sin\varphi(t) + \frac{\ddot\varphi(t+\tau) - \ddot\varphi(t-\tau)}{2\tau}\rho(t)\cos\varphi(t) \approx$$
$$\approx \dddot\rho\sin\varphi + \dddot\varphi\rho\cos\varphi\ ;$$
$$\left\langle \vec u^{(2)}(t), \frac{\dot{\vec u}^{(2)}(t+\tau) - \dot{\vec u}^{(2)}(t-\tau)}{2\tau}\right\rangle =$$
$$= \left(\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi\right)\left(\frac{\ddot\rho(t+\tau) - \ddot\rho(t-\tau)}{2\tau}\cos\varphi - \frac{\ddot\varphi(t+\tau) - \ddot\varphi(t-\tau)}{2\tau}\rho\sin\varphi\right) +$$
$$+ \left(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi\right)\left(\frac{\ddot\rho(t+\tau) - \ddot\rho(t-\tau)}{2\tau}\sin\varphi + \frac{\ddot\varphi(t+\tau) - \ddot\varphi(t-\tau)}{2\tau}\rho\cos\varphi\right) =$$
$$= \frac{\ddot\rho(t+\tau) - \ddot\rho(t-\tau)}{2\tau}\dot\rho + \frac{\ddot\varphi(t+\tau) - \ddot\varphi(t-\tau)}{2\tau}\rho^2\dot\varphi \approx\ ;$$
$$\approx \dddot\rho\dot\rho + \dddot\varphi\rho^2\dot\varphi$$
$$u_2^{(2)}(t)\left\langle \vec u^{(2)}(t), \frac{\dot{\vec u}^{(2)}(t+\tau) - \dot{\vec u}^{(2)}(t-\tau)}{2\tau}\right\rangle + {\Delta_2}^2\ddot u_2^{(2)}(t) =$$
$$= (\dot\rho\cos\varphi - \rho\dot\varphi\sin\varphi)(\dddot\rho\dot\rho + \dddot\varphi\rho^2\dot\varphi) +$$
$$+ (c^2 - \dot\rho^2 - \rho^2\dot\varphi^2)(\dddot\rho\cos\varphi - \dddot\varphi\rho\sin\varphi) =$$
$$= (\dddot\varphi\rho^2\dot\rho\dot\varphi + c^2\dddot\rho - \dddot\rho\rho^2\dot\varphi^2)\cos\varphi +\ ;$$
$$+ (\dddot\varphi\rho\dot\rho^2 - \dddot\rho\dot\rho\rho\dot\varphi - c^2\dddot\varphi\rho)\sin\varphi$$
$$u_3^{(2)}(t)\left\langle \vec u^{(2)}(t), \frac{\dot{\vec u}^{(2)}(t+\tau) - \dot{\vec u}^{(2)}(t-\tau)}{2\tau}\right\rangle + {\Delta_2}^2\ddot u_3^{(2)}(t) =$$
$$(\dot\rho\sin\varphi + \rho\dot\varphi\cos\varphi)(\dddot\rho\dot\rho + \dddot\varphi\rho^2\dot\varphi) +$$
$$+ (c^2 - \dot\rho^2 - \rho^2\dot\varphi^2)(\dddot\rho\sin\varphi + \dddot\varphi\rho\cos\varphi) =$$
$$= (\dddot\varphi\rho^2\dot\rho\dot\varphi + \dddot\rho c^2 - \dddot\rho\rho^2\dot\varphi^2)\sin\varphi +$$
$$+ (\dddot\rho\dot\rho\rho\dot\varphi + \dddot\varphi c^2\rho - \dddot\varphi\rho\dot\rho^2)\cos\varphi;$$
$$G_2^{(2)rad} = -\frac{e_2^2}{m_2c^2}\frac{u_2^{(2)}(t)\left\langle \vec u^{(2)}(t), \ddot{\vec u}^{(2)}(t)\right\rangle + {\Delta_2}^2\ddot u_2^{(2)}(t)}{{\Delta_2}^3} =$$
$$= -\frac{e_2^2}{m_2c^2}\frac{(\dddot\varphi\rho^2\dot\rho\dot\varphi + c^2\dddot\rho - \dddot\rho\rho^2\dot\varphi^2)\cos\varphi + (\dddot\varphi\rho\dot\rho^2 - \dddot\rho\dot\rho\rho\dot\varphi - c^2\dddot\varphi\rho)\sin\varphi}{{\Delta_2}^3};$$

$$G_3^{(2)rad} = -\frac{e_2^2}{m_2c^2}\frac{u_3^{(2)}(t)\left\langle \vec{u}^{(2)}(t), \ddot{\vec{u}}^{(2)}(t)\right\rangle + \Delta_2{}^2 \ddot{u}_3^{(2)}(t)}{\Delta_2{}^3} =$$

$$= -\frac{e_2^2}{m_2c^2}\frac{(\ddot\varphi\rho^2\dot\rho\dot\varphi + c^2\ddot\rho - \ddot\rho\rho^2\dot\varphi^2)\sin\varphi + (\ddot\rho\dot\rho\rho\dot\varphi + c^2\ddot\varphi\rho - \ddot\varphi\rho\dot\rho^2)\cos\varphi}{\Delta_2{}^3};$$

$$R_r^{rad}(\rho,\varphi) = G_2^{(2)rad}\cos\varphi + G_3^{(2)rad}\sin\varphi =$$

$$= -\frac{e_2^2}{m_2c^2}\frac{\ddot\varphi\rho^2\dot\rho\dot\varphi + (c^2-\rho^2\dot\varphi^2)\ddot\rho}{\Delta_2{}^3};$$

$$R_\phi^{rad}(\rho,\varphi) = G_3^{(2)rad}\cos\varphi - G_2^{(2)rad}\sin\varphi =$$

$$= -\frac{e_2^2}{m_2c^2}\frac{\ddot\rho\dot\rho\rho\dot\varphi + \rho(c^2-\dot\rho^2)\ddot\varphi}{\Delta_2{}^3}.$$

Finally, we have to study the system with unknown functions $\rho(t), \varphi(t)$ for $t \ge 0$:

$$\ddot\rho = \rho\dot\varphi^2 + \frac{e_1e_2}{m_2c}\frac{\sqrt{c^2-\dot\rho^2-\rho^2\dot\varphi^2}}{\rho^2}\frac{c^2-\dot\rho^2}{c^2} - $$
$$-\frac{e_2^2}{m_2c^2}\frac{\ddot\varphi\rho^2\dot\rho\dot\varphi + (c^2-\rho^2\dot\varphi^2)\ddot\rho}{\Delta_2{}^3},\ t \ge 0;$$
$$\ddot\varphi = -\frac{2\dot\rho\dot\varphi}{\rho} - \frac{e_1e_2}{m_2c}\frac{\dot\rho\dot\varphi\sqrt{c^2-\dot\rho^2-\rho^2\dot\varphi^2}}{\rho^2}\frac{c^2-\dot\rho^2}{c} - \qquad (5)$$
$$-\frac{e_2^2}{m_2c^2}\frac{\ddot\rho\dot\rho\rho\dot\varphi + \rho(c^2-\dot\rho^2)\ddot\varphi}{\Delta_2{}^3},\ t \ge 0.$$

III. TRANSITION FROM A STATIONARY STATE WITH PRESCRIBED RADIUS TO A STATIONARY STATE WITH LARGER RADIUS

Let us set $\dot\rho = r$; $\dot\varphi = \phi$.

Then $\rho(t) = \rho_0 + \int_0^t r(s)ds$; $\varphi(t) = \varphi_0 + \int_0^t \phi(s)ds$.

Reduce the above system (5) to the following one:

$$\dot r = \rho\phi^2 + \frac{e_1e_2}{m_2c}\frac{\sqrt{c^2-r^2-\rho^2\phi^2}}{\rho^2}\frac{c^2-r^2}{c} - $$
$$-\frac{e_2^2}{m_2c^2}\frac{\ddot\phi\rho^2 r\phi + (c^2-\rho^2\phi^2)\ddot r}{\Delta_2{}^3} \equiv F_r;$$
$$\dot\phi = -\frac{2r\phi}{\rho} - \frac{e_1e_2}{m_2c}\frac{r\phi\sqrt{c^2-r^2-\rho^2\phi^2}}{\rho^2}\frac{c^2-r^2}{c} - \qquad (6)$$
$$-\frac{e_2^2}{m_2c^2}\frac{\ddot r r\rho\phi + \rho(c^2-r^2)\ddot\phi}{\Delta_2{}^3} \equiv F_\phi.$$

We decompose the right-hand sides into the Lorentz part and radiation part, namely

$$L_r = \rho\phi^2 - \frac{|e_1e_2|}{m_2c^3}\frac{\left(c^2-r^2\right)\sqrt{c^2-r^2-\rho^2\phi^2}}{\rho^2};$$

$$L_\phi = -\frac{2r\phi}{\rho} + \frac{|e_1e_2|}{m_2c^3}\frac{r\phi\sqrt{c^2-r^2-\rho^2\phi^2}}{\rho^2};$$

$$L_r^{rad} = -\frac{e_2^2}{m_2c^2}\frac{\ddot r r\rho\phi + \rho(c^2-r^2)\ddot\phi}{\Delta_2{}^3};$$

$$L_\phi^{rad} = -\frac{e_2^2}{m_2c^2}\frac{\ddot r r\rho\phi + \rho(c^2-r^2)\ddot\phi}{\Delta_2{}^3}.$$

We look for a periodic solution (r,ϕ) of (6) such that $\rho(t)$ to be periodic function and $\varphi(t)$ to be unbounded function.

Denote by $C_T^\infty[0,\infty)$ the set of all infinitely differentiable *T*-periodic functions such that $r(kT)=0\ (k=0,1,\ldots)$.

Let $k_0,k_1 \in N,\ 0<k_0<k_1$. From $C_T^\infty[0,\infty)$ we obtain the following set of functions: we redefine every function $r(.) \in C_T^\infty[0,\infty)$ in the following way:

$$\tilde r(t) = \begin{cases} r(t),\ t\in[0,k_0T), \\ r_1 + r(t),\ t\in[k_0T,k_1T], \\ r(t),\ \ t\in(k_1T,\infty) \end{cases}$$

where r_1 is a positive constant and $r_1 + r(t) \ge 0$.

The function $\tilde r(t)$ is not continuous on $[0,\infty)$ with jump r_1 at k_0T, k_1T. We can smooth out these functions using some technique from the Sobolev-Schwartz distribution theory [18]-[20] (cf. also [21], [22]).

Lemma 1 [22] For every compact set E and every open set F containing E there is an infinitely differentiable with compact support test function $\eta(t)$ such that $\eta(t)=1$, $t\in E,\ \eta(t)=0,\ t\notin F$ and $0\le\eta(t)\le1$ for the other values of t.

Then we can define the function $\eta(.)\in C^\infty$ by the formulas

$$\eta(t) = \begin{cases} 0,\ t\le k_0T \\ 0\le\eta(t)\le1,\ t\in[k_0T,k_0T+\varepsilon] \\ 1,\ t\in[k_0T+\varepsilon,k_1T-\varepsilon] \\ 0\le\eta(t)\le1, t\in[k_1T-\varepsilon,k_1T] \\ 0,\ t\ge k_1T \end{cases}, \text{ where } 0<\varepsilon<T.$$

For every $r(.)\in C_T^\infty[0,\infty)$ we define the function

$$\tilde r(t) = \begin{cases} r(t),\ t\in[0,k_0T], r(k_0T)=0 \\ \eta(t)(r_1+r(t)),\ t\in(k_0T,k_1T) \\ r(t),\ \ t\in[k_1T,\infty),\ r(k_1T)=0. \end{cases}$$

One can verify that $\tilde r(t)$ is infinitely smooth function. In this way we obtain a set of infinitely smooth functions $\tilde r(t)$.

Remark 1. These new function we denote again by $r(t)$. It is *T*-periodic on $[0,k_0T]\cup[k_1T,\infty)$. On $[k_0T,k_1T]$ it is *T*-

periodic too but raised with r_1 and at k_0T, k_1T the jumps are smooth out. The obtained function set we denote by $C_T^{\infty}[0,\infty)$.

Now we introduce the sets $(m=0,1,2,...)$:

$$M_r=\left\{r(.)\in C_T^{\infty}[0,\infty):\ \left|\frac{d^m r(t)}{dt^m}\right|\le \omega^m R_0 e^{\mu(t-kT)};\right.$$

$$\int_{kT}^{(k+1)T} r(s)ds=0,(k=0,1,...,k_0-1,\ k_1,...);$$

$$\int_{kT}^{(k+1)T} r(s)ds=T\,r_1,(k=k_0+1,k_0+2,...,k_1-2);$$

$$\left.\left|\frac{d^m r(t)}{dt^m}\right|\le \omega^m (r_1+R_0)e^{\mu(t-kT)};\ \int_{k_0T}^{(k_0+1)T} r(s)ds\le Tr_1;\ \int_{(k_1-1)T}^{k_1T} r(s)ds\le Tr_1\right\}$$

and

$$M_\phi=\left\{\phi\in C_T^{\infty}[0,\infty):\left|\frac{d^m\phi(t)}{dt^m}\right|\le \omega^m\Phi_0 e^{\mu(t-kT)};\ \int_{kT}^{(k+1)T}\phi(t)dt=T\phi_0\right\},$$

$$(k,m=0,1,2,\ldots). \tag{8}$$

Remark 2. We choose the constants r_1, R_0 such that $|r(t)|\le (r_1+R_0)e^{\mu T}\le \bar{c}<c$ and further on we will again denote functions without a tilde.

The Cartesian product $M_r\times M_\phi$ can be endowed with a saturated family of pseudometrics

$$d_{(m,k)}((r,\phi),(\bar r,\bar\phi))=d_{(m,k)}(r,\bar r)+d_{(m,k)}(\phi,\bar\phi)$$

where

$$d_{(m,k)}(r,\bar r)=\sup\left\{\left|\frac{d^m r(t)}{dt^m}-\frac{d^m\bar r(t)}{dt^m}\right|e^{-\mu(t-kT)}:\ t\in[kT;(k+1)T]\right\};$$

$$d_{(m,k)}(\phi,\bar\phi)=\sup\left\{\left|\frac{d^m\phi(t)}{dt^m}-\frac{d^m\bar\phi(t)}{dt^m}\right|e^{-\mu(t-kT)}:t\in[kT,(k+1)T]\right\}$$

$(k,m=0,1,2,...)$.

Let $(r,\phi)\in M_r\times M_\phi$. Prior to expose the mathematical results we want to show what we achieve from physical point of view.

1) The distance function $\rho(t)$ is T-periodic one on $t\in[0;\,k_0T]$. If $t\in(0;\,\hat kT)$ for some $\hat k\in\{1,2,...,k_0\}$ then

$$\rho(t)=\rho_0+\int_0^t\tilde r(s)ds=\rho_0+\sum_{k=0}^{\hat k}\int_{(k-1)T}^{kT}\tilde r(s)ds+\int_{\hat kT}^{t}\tilde r(s)ds\Rightarrow$$

$$\rho(t)\le\rho_0+R_0(e^{\mu(t-\hat kT)}-1)/\mu\le\rho_0+R_0(e^{\mu T}-1)/\mu\equiv\breve\rho;$$

$$\rho(t)\ge\rho_0-R_0(e^{\mu T}-1)/\mu\equiv\widehat\rho>0.$$

2) The distance function increases for $t\in(k_0T,k_1T)$, because $r_1+r(t)\ge 0$.

Besides denoting by $\rho_1=\rho_0+(k_1-k_0)T$ we obtain

$$\rho(k_1T)=\rho_0+\int_0^{k_1T}r(s)ds=\rho_0+\int_0^T r(s)ds...+$$

$$+\int_{(k_0-1)T}^{k_0T}r(s)ds+\int_{k_0T}^{(k_0+1)T}r(s)ds+...+\int_{(k_1-1)T}^{k_1T}r(s)ds=$$

$$=\rho_0+\int_{k_0T}^{(k_0+1)T}r(s)ds+...+\int_{(k_1-1)T}^{k_1T}r(s)ds\le\rho_0+(k_1-k_0)Tr_1=\rho_1.$$

For $t\in(\,k_1T,(k_1+1)T)$ we obtain

$$\rho(t)=\rho_0+\int_0^{k_1T}r(s)ds+\int_{k_1T}^{t}r(s)ds=\rho_1+\int_{k_1T}^{t}r(s)ds.$$

Therefore the function $\hat\rho(t)=\int_{k_1T}^{t}r(s)ds$ is T-periodic one because $\int_{kT}^{(k+1)T}r(s)ds=0$ for $k=k_1,k_1+1,...$.

It is easy to check that

$$0<\widehat\rho\equiv\rho_0-R_0\frac{e^{\mu T}-1}{\mu}\le\rho(t)=\rho_0+\int_0^t r(s)ds\le\rho_0+R_0\frac{e^{\mu T}-1}{\mu}\equiv\breve\rho.$$

For the polar angle we have

$$\lim_{t\to\infty}\varphi(t)=\varphi_0+\int_0^\infty\phi(s)ds=\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\phi(s)ds=T\sum_{k=0}^{\infty}\phi_0=\infty.$$

In this manner we demonstrate that the electron jumps (but continuously) from a level with radius $\rho(.)\in[\widehat\rho,\breve\rho]$ to the level with radius $\rho(.)\in[\widehat\rho_1,\breve\rho_1]$. The trajectory does exist and these are the functions $\rho(t),\varphi(t)$ defined on the interval $[0,\infty)$. It remains to prove an existence-uniqueness of solution of (6) belonging to $M_r\times M_\phi$ and then $\rho(t)$ is periodic function (and therefore bounded) for $t\in[0;\,k_0T]$, increasing for $t\in[\,k_0T;\,k_1T]$ and again periodic one for $t\in[\,k_1T;\,\infty)$. In this manner we describe the process of transition of an electron from one Bohr energy level to the second one with a larger radius. We conclude that the electron has a deterministic orbit even jumping from one level to another one. It remains to show that such trajectories do exists.

IV. AUXILLIARY ASSERTIONS

Lemma 2. The set $M_r\times M_\phi$ is closed.

The proof is given in Appendix A.

Define an operator $B=(B_r,B_\phi):M_r\times M_\phi\to M_r\times M_\phi$ for $t\in[kT,(k+1)T],r_1>0$ by the formulas:

$$B_{r,k}(r,\phi)(t):=\int_{kT}^{t}F_r(r,\phi)(s)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-$$

$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(p)dpds,\ (k=0,1,...,k_0-1);$$

$$B_{r,k}(r,\phi)(t):=\eta(t)\left(r_1+\int_{kT}^{t}F_r(r,\phi)(s)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\right.$$

$$\left.-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(p)dpds\right)\ (k=k_0,...,k_1-1)\ ;$$

$$B_{r,k}(r,\phi)(t):=$$

$$\int_{kT}^{t}F_r(r,\phi)(s)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds$$

$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(p)dpds,\quad (k=k_1,k_1+1,...)\ ;\tag{9}$$

and for $t\in[kT,(k+1)T],(k=0,1,...);\ \phi_0>0$ by the formula:

$$B_{\phi,k}(r,\phi)(t):=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)(s)ds-$$

$$\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_\phi(r,\phi)(p)dpds,\tag{10}$$

Lemma 3. If $r^2+\rho^2\phi^2\le R_0{}^2e^{2\mu T}+\breve{\rho}_1{}^2\Phi_0{}^2e^{2\mu T}\le\bar{c}^2$ and $r_1<R_0$, then the radiation terms satisfy the inequalities

$$\left|L_r^{rad}\right|\le\frac{e_2^2}{m_2c^2}\left|\frac{\ddot{r}r\rho\phi+\rho(c^2-r^2)\ddot{\phi}}{\Delta_2{}^3}\right|\le\frac{e_2^2}{m_2}\frac{1+\chi^n/\mu+\bar{c}}{c^3(1-\beta^2)^{3/2}}\chi^ne^{\mu T}$$

$$\left|L_\phi^{rad}\right|\le\frac{e_2^2}{m_2c^2}\left|\frac{\ddot{r}r\rho\phi+\rho(c^2-r^2)\ddot{\phi}}{\Delta_2{}^3}\right|\le\frac{e_2^2}{m_2}\frac{1+\chi^n/\mu+\bar{c}}{c^3(1-\beta^2)^{3/2}}\chi^ne^{\mu T}$$

where $\omega/\mu=\chi<1$ and $\beta=\bar{c}/c<1$ for some positive integer n.

The proof is implied by the inequalities given in Appendix B.

Lemma 4 (Main Lemma) The periodic problem (6) has a solution $(r,\phi)\in M_r\times M_\phi$ iff the operator B has a fixed point belonging to $M_r\times M_\phi$.

Proof: Let us assume that (6) has a solution.

If $r_1=0$ the proof can be acomplished as in [5]. We consider the case $t=(k+1)T=r_1,\ (k=k_0+1,k_0+2,...,k_1-1)$. After integration of (6) we have

$$r(t)=r_1+\int_{kT}^{t}F_\phi(r,\phi)(s)ds\ ,\ t\in[kT,(k+1)T];\ (k=k_0+1,...,k_1-2)\ ;$$

$$\phi(t)=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)(s)ds\ ,\ t\in[kT,(k+1)T]\ ,\ (k=0,1,2,\ldots).$$

Substituting $t=(k+1)T$ in

$$r(t)=r_1+\int_{kT}^{t}F_\phi(r,\phi)(s)ds\ ,\ t\in[kT,(k+1)T]$$

we obtain $r_1=r_1+\int_{kT}^{(k+1)T}F_r(r,\phi)ds\Rightarrow\int_{kT}^{(k+1)T}F_r(r,\phi)ds\ =0$.

In a similar way we have

$$0=\phi((k+1)T)=\phi(kT)+\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds\Rightarrow\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds=0.$$

Therefore,

$$r(t)=r_1+\int_{kT}^{t}F_r(r,\phi)ds\Leftrightarrow$$

$$r(t)=r_1+\int_{kT}^{t}F_r(r,\phi)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds$$

and

$$\phi(t)=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)ds\Leftrightarrow$$

$$\phi(t)=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)ds\ -\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds\ .$$

Changing the order of integration and taking into account $r\big((k+1)T\big)=r(kT)=r_1$ and $\int_{kT}^{(k+1)T}r(s)ds=r_1T$ we have

$$\int_{kT}^{(k+1)T}\int_{kT}^{p}F_r(r,\phi)(s)dsdp=\int_{kT}^{(k+1)T}\int_{s}^{(k+1)T}F_r(r,\phi)(s)dpds=$$

$$=\int_{kT}^{(k+1)T}\big((k+1)T-s\big)F_r(r,\phi)(s)ds=$$

$$=(k+1)T\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\int_{kT}^{(k+1)T}sF_r(r,\phi)(s)ds=$$

$$-\int_{kT}^{(k+1)T}sF_r(r,\phi)(s)ds=-\int_{kT}^{(k+1)T}s\frac{dr(s)}{ds}ds=-\int_{kT}^{(k+1)T}s\,dr(s)=$$

$$-(k+1)T\,r\big((k+1)T\big)+kT\,r(kT)+\int_{kT}^{(k+1)T}r(s)ds=-r_1T+r_1T=0$$

Consequently

$$r(t)=r_1+\int_{kT}^{t}F_r(r,\phi)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds$$

can be written in the form

$$r(t)=r_1+\int_{kT}^{t}F_r(r,\phi)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-$$
$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_r(r,\phi)(s)dsdp\ .$$

In view of the definition of $\eta(t)$ we can multiply the above equality by $\eta(t)$ on $[k_0T,k_1T]$, that is,

$$r(t)=\eta(t)\Bigg(r_1+\int_{kT}^{t}F_r(r,\phi)ds-$$
$$-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_r(r,\phi)(s)dsdp\Bigg)$$

($k=k_0,...,k_1-1$).

The last equality means that $r=B_r(r,\phi)$.

For the second component in asimilar way we obtain

$$\int_{kT}^{(k+1)T}\int_{kT}^{p}F_\phi(r,\phi)(s)dsdp=\int_{kT}^{(k+1)T}\int_{s}^{(k+1)T}F_\phi(r,\phi)(s)dpds=$$
$$=\int_{kT}^{(k+1)T}((k+1)T-s)F_\phi(r,\phi)(s)ds=-T\phi_0+T\phi_0=0.$$

Therefore,

$$\phi(t)=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds-$$
$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_\phi(r,\phi)(s)dsdp$$

which means $\phi=B_\phi(r,\phi)$, i.e. (r,ϕ) is a fixed point of the system $(r,\phi)=(B_r(r,\phi),B_\phi(r,\phi))$.

Conversely, let $(r,\phi)\in M_r\times M_\phi$ be a fixed point of *B*. Then substituting $t=kT$ in

$$r(t)=\eta(t)\Bigg(r_1+\int_{kT}^{t}F_r(r,\phi)ds-$$
$$-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_r(r,\phi)(s)dsdp\Bigg)$$

we obtain

$$r(kT)=r_1+\int_{kT}^{kT}F_r(r,\phi)ds-$$
$$-\left(\frac{kT-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(\theta)d\theta ds$$
$$\Rightarrow 0=\frac{1}{2}\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(\theta)d\theta ds\ .$$

Consequently

$$\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds=0\Rightarrow\int_{kT}^{(k+1)T}\int_{kT}^{s}F_r(r,\phi)(\theta)d\theta ds=0\ .$$

If we suppose that $\left|\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds\right|=\delta\neq 0$ then

$$\left|\int_{kT}^{(k+1)T}F_r(r,\phi)(s)ds\right|\le$$
$$\le\left(\bar{c}\chi^m\Phi_0+\frac{|e_1e_2|}{m_2}\frac{1}{\widehat{\rho}^2}+\frac{e_2^2}{m_2}\frac{(r_1+\dfrac{\chi^nR_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}\chi^ne^{2\mu T}\right)\frac{e^{\mu T}-1}{\mu}.$$

The last term becames smaller than δ for sufficiently large $\mu>0$ and $\mu T=const.$

Consequently $r(t)=r_1+\int_{kT}^{t}F_r(r,\phi)ds\Rightarrow\dot{r}(t)=F_r(r,\phi)(t)$.

In the second component of *B* we put $t=kT$ and get

$$\phi(t)=\phi_0+\int_{kT}^{t}F_\phi(r,\phi)ds-\left(\frac{t-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds-$$
$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_\phi(r,\phi)(s)dsdp\ .$$

Then we have

$$\phi(kT)=\phi_0+\int_{kT}^{kT}F_\phi(r,\phi)ds-\left(\frac{kT-kT}{T}-\frac{1}{2}\right)\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds-$$
$$-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{p}F_\phi(r,\phi)(s)dsdp\ ,$$

and in view of $\phi(kT)=\phi_0$ $(k=0,1,2,...)$ we get

$$0=\frac{1}{2}\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds-\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_\phi(r,\phi)(\theta)d\theta ds\ .$$

Consequently

$$\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds=0\Rightarrow\frac{1}{T}\int_{kT}^{(k+1)T}\int_{kT}^{s}F_\phi(r,\phi)(\theta)d\theta ds=0\ .$$

If we assume that $\left|\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds\right|=\delta>0$ then using the estimate of the radition term from Lemma 3 we have

$$\left|\int_{kT}^{(k+1)T}F_\phi(r,\phi)(s)ds\right|\le$$
$$\le\left(\frac{2\chi^{2m}\bar{c}\Phi_0}{\widehat{\rho}}+\frac{|e_1e_2|}{m_2}\chi^m\left(\frac{\Phi_0}{c\widehat{\rho}^2}+\frac{1+\chi^m/\mu+\bar{c}}{c^3(1-\beta^2)^{3/2}}e^{\mu T}\right)\right)\frac{e^{\mu T}}{\mu},$$

where $\chi=\omega/\mu<1$ and then for sufficiently large μ and

$\mu T = const.$ we obtain a contradiction. Differentiating $\phi(t)=\phi_0+\int\limits_{kT}^{t}F_\phi(r,\phi)ds$ we obtain $\dot{\phi}(t)=F_\phi(r,\phi)$.

The Main Lemma is thus proved.

V. EXISTENCE-UNIQUENESS THEOREM

Theorem 1. Let the following conditions be fulfilled:

$$r_1+\left(\bar{c}\Phi_0+\frac{|e_1e_2|}{m_2}\frac{1}{\hat{\rho}^2}+\frac{e_2^2}{m_2}\frac{(r_1+\frac{\chi^m R_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}\chi^m e^{2\mu T}\right)\frac{1}{\mu}\le R_0;$$

$$\phi_0+\left(\frac{2}{\hat{\rho}}+\frac{|e_1e_2|}{m_2}\frac{c}{\hat{\rho}^2}+\frac{e_2^2}{m_2}\frac{(r_1+\frac{\chi^m R_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}e^{2\mu T}\right)\frac{\chi^m e^{2\mu T}}{\mu}\le \Phi_0$$

for some positive integer *m*.

Then there exists a unique solution belonging to $M=M_r\times M_\phi$.

***Proof*:** Define an operator $B=(B_r,B_\phi):M_r\times M_\phi\to M_r\times M_\phi$ by the above formulas (9).

First we show that B maps $M=M_r\times M_\phi$ into itself. Indeed, the optator B is defined in such away that the operator functions $B=(B_r,B_\phi)\in\tilde{C}_T^{\infty}[0,\infty)$. It remains to show the inequalities for $t\in[kT,(k+1)T], k=k_0,...,k_1$:

$$\left|B_{r,k}(r,\phi)(t)\right|\le r_1+\left|\int\limits_{kT}^{t}F_r(r,\phi)(s)ds\right|ds\le$$

$$\le r_1+\left(\bar{c}\Phi_0+\frac{|e_1e_2|}{m_2}\frac{1}{\hat{\rho}^2}+\frac{e_2^2}{m_2}\frac{(r_1+\frac{\chi^m R_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}\chi^m e^{2\mu T}\right)\frac{e^{\mu(t-kT)}}{\mu}\le$$

$$\le R_0e^{\mu(t-kT)}$$

For $t\notin[kT,(k+1)T], k=k_0,...,k_1$ the summand r_1 misses.

For $t\in[kT,(k+1)T],(k=0,1,...)$ we have

$$\left|B_{\phi,k}(r,\phi)(t)\right|\le\phi_0+\left|\int\limits_{kT}^{t}F_\phi(r,\phi)(s)ds\right|\le\phi_0+$$

$$+\left(\left(\frac{2}{\hat{\rho}}+\frac{|e_1e_2|}{m_2}\frac{c}{\hat{\rho}^2}\right)\chi^{2m}+\frac{e_2^2}{m_2}\frac{(r_1+\frac{\chi^m R_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}\chi^m e^{2\mu T}\right)e^{2\mu T}\frac{e^{\mu(t-kT)}}{\mu}\le$$

$$\le\phi_0+\left(\frac{2}{\hat{\rho}}+\frac{|e_1e_2|}{m_2}\frac{c}{\hat{\rho}^2}+\frac{e_2^2}{m_2}\frac{(r_1+\frac{\chi^m R_0}{\mu})\bar{c}+c^2\breve{\rho}_1\Phi_0}{c^5(1-\beta^2)^{3/2}}e^{2\mu T}\right)\chi^m e^{2\mu T}\frac{e^{\mu(t-kT)}}{\mu}\le$$

$$\le\Phi_0e^{\mu(t-kT)}.$$

Further on for $t\in[kT,(k+1)T],\ k=k_0,...,k_1$ one has $\int\limits_{kT}^{(k+1)T}B_{r,k}(r,\phi)(s)ds=r_1T>0$, because $\int\limits_{kT}^{(k+1)T}\left(\frac{t-kT}{T}-\frac{1}{2}\right)dt=0$.

For $t\notin[kT,(k+1)T],\ k=k_0,...,k_1$ we have

$$\int\limits_{kT}^{(k+1)T}B_{r,k}(r,\phi)(s)ds=0.$$

For $t\in[kT,(k+1)T],k=0,1,...$ it follows

$$\int\limits_{kT}^{(k+1)T}B_{\phi,k}(r,\phi)(s)ds=T\phi_0.$$

We show that the operator B is contractive one in the sense of [23].

In view of Appendix C and $r_1+\chi^m R_0e^{\mu T}\le\bar{c}$ we have

$$\left|B_{r,k}(r,\phi)(t)-B_{r,k}(\bar{r},\bar{\phi})(t)\right|\le$$

$$\le\int\limits_{kT}^{t}\left(\left|\frac{\partial F_r}{\partial\rho}\right|\left|\rho(s)-\bar{\rho}(s)\right|+\left|\frac{\partial F_r}{\partial r}\right|\left|r(s)-\bar{r}(s)\right|+\left|\frac{\partial F_r}{\partial\phi}\right|\left|\phi(s)-\bar{\phi}(s)\right|\right)ds\le$$

$$\le e^{\mu(t-kT)}K_{(m,k)}(r)d_{(m,k)}\left((r,\phi),(\bar{r},\bar{\phi})\right)$$

where for some *m*

$$K_{(m,k)}(r)=\frac{1}{\mu^m}\left[\frac{\left(\phi_0+\chi^m\Phi_0e^{\mu T}\right)^2}{\mu^2}+\frac{2c}{\mu}+\frac{3}{\mu\hat{\rho}^2}+\right.$$

$$+\frac{|e_1e_2|}{m_2(1-\beta^2)^{5/2}}\left(\frac{5c}{\mu^2\hat{\rho}^3}+\frac{5\omega^2\Phi_0}{\mu^2}\frac{2\chi^m e^{\mu T}+1}{c^3}+\right.$$

$$\left.\left.+\frac{c^2(1+4\omega^2\chi^m+3\omega^2)}{\mu}+\Phi_0\frac{1+\chi^m e^{\mu T}}{\mu}+\frac{11\omega^2\chi^m\breve{\rho}}{\mu c^2}\right)\right]\le$$

$$\le K<1/2.$$

It follows

$$d_{(0,k)}\left(B_{r,k}(r,\phi),B_{r,k}(\bar{r},\bar{\phi})\right)\le K_{(m,k)}(r)\,d_{(m,k)}\left((r,\phi),(\bar{r},\bar{\phi})\right).$$

For the second component we have

$$\left|B_{\phi,k}(r,\phi)(t)-B_{\phi,k}(\bar{r},\bar{\phi})(t)\right|\le$$

$$\le\int\limits_{kT}^{t}\left(\left|\frac{\partial F_\phi}{\partial\rho}\right|\left|\rho(s)-\bar{\rho}(s)\right|+\left|\frac{\partial F_\phi}{\partial r}\right|\left|r(s)-\bar{r}(s)\right|+\left|\frac{\partial F_\phi}{\partial\phi}\right|\left|\phi(s)-\bar{\phi}(s)\right|\right)ds\le$$

$$\le e^{\mu(t-kT)}K_{(m,k)}(\phi)d_{(m,k)}\left((r,\phi),(\bar{r},\bar{\phi})\right).$$

It follows

$$d_{(0,k)}\left(B_{\phi,k}(r,\phi),B_{\phi,k}(\bar{r},\bar{\phi})\right)\le K_\phi\,d_{(n,k)}\left((r,\phi),(\bar{r},\bar{\phi})\right),$$

where for some *m*.

$$K_{(m,k)}(\phi)=\frac{1}{\mu^m}\left[\frac{2\bar{c}\chi^m\Phi_0e^{\mu T}}{\mu\hat{\rho}^2}+\frac{2\phi_0+2\chi^m\Phi_0e^{\mu T}+2\bar{c}}{\hat{\rho}}+\right.$$

$$\frac{|e_1e_2|}{m_2}\left(\frac{2c\bar{c}\chi^m\Phi_0 e^{\mu T}}{\mu\bar{\rho}^3(1-\beta^2)^{1/2}}+4\omega^2\chi^m\Phi_0\frac{2+\chi^m e^{\mu T}}{\mu c^3(1-\beta^2)^{5/2}}+\right.$$
$$+\frac{c\phi_0+c\chi^m\Phi_0 e^{\mu T}}{\bar{\rho}^2(1-\beta^2)^{1/2}}+\frac{\omega^2\chi^m(\breve{\rho}+2\bar{c}\breve{\rho}+3c)+3ce^{\mu T}}{c^4(1-\beta^2)^{5/2}}+$$
$$\left.\left.+\frac{\bar{c}c}{\bar{\rho}^2}+\frac{(\phi_0+\chi^m\Phi_0 e^{\mu T})^2}{(1-\beta^2)^{1/2}}+\frac{5\omega^2\chi^m\breve{\rho}}{c(1-\beta^2)^{5/2}}\right)\right]\le K<1/2\,.$$

Denoting by $\bar{B}_{r,k}=B_{r,k}(\bar{r},\bar{\phi})$, $\bar{B}_{\phi,k}=B_{\phi,k}(\bar{r},\bar{\phi})$ we obtain

$$d_{(0,k)}\big((B_{r,k},B_{\phi,k}),(\bar{B}_{r,k},\bar{B}_{\phi,k})\big)\le$$
$$\le\big(K_{(m,k)}(r)+K_{(m,k)}(\phi)\big)d_{(m,k)}\big((r,\phi),(\bar{r},\bar{\phi})\big)\le Kd_{(m,k)}\big((r,\phi),(\bar{r},\bar{\phi})\big)$$

We note that the contractive constants $K_{(m,k)}(r),K_{(m,k)}(\phi)$ do not depend on k.

For the derivatives we proceed in a similar way taking into account

$$\left|\frac{dB_{r,k}(r,\phi)(t)}{dt}-\frac{dB_{r,k}(\bar{r},\bar{\phi})(t)}{dt}\right|\le$$
$$\le\left|\frac{\partial F_r}{\partial\rho}\right||\rho(s)-\bar{\rho}(s)|+\left|\frac{\partial F_r}{\partial r}\right||r(s)-\bar{r}(s)|+\left|\frac{\partial F_r}{\partial\phi}\right||\phi(s)-\bar{\phi}(s)|\le$$
$$\le e^{\mu(t-kT)}K_{(1+m,k)}(r)d_{(1+m,k)}(r,\bar{r})$$

that implies

$$d_{(1,k)}(B_{r,k}(r,\phi),B_{r,k}(\bar{r},\bar{\phi}))\le K_{(1+m,k)}(r)d_{(1+m,k)}\big((r,\phi),(\bar{r},\bar{\phi})\big).$$

For the second component we have

$$\left|\frac{dB_{\phi,k}(r,\phi)(t)}{dt}-\frac{dB_{\phi,k}(\bar{r},\bar{\phi})(t)}{dt}\right|\le$$
$$\le\left|\frac{\partial F_\phi}{\partial\rho}\right||\rho(s)-\bar{\rho}(s)|+\left|\frac{\partial F_\phi}{\partial r}\right||r(s)-\bar{r}(s)|+\left|\frac{\partial F_\phi}{\partial\phi}\right||\phi(s)-\bar{\phi}(s)|\le$$
$$\le e^{\mu(t-kT)}K_{(1+m,k)}(\phi)d_{(1+m,k)}(\phi,\bar{\phi})$$

that implies

$$d_{(1,k)}(B_{\phi,k}(r,\phi),B_{\phi,k}(\bar{r},\bar{\phi}))\le$$
$$\le\big(K_{(1+m,k)}(r)+K_{(1+m,k)}(\phi)\big)d_{(k,1+m)}\big((r,\phi),(\bar{r},\bar{\phi})\big)\le$$
$$\le K\,d_{(1+m,k)}\big((r,\phi),(\bar{r},\bar{\phi})\big).$$

The number m is chosen sufficiently large to imply $K_{(1+m,k)}(r)+K_{(1+m,k)}(\phi)\le 2K<1$ as the highest degree of ω is 3. Continuing in this way, we notice that the degree of ω is with two higher than thr degree of μ (cf. [3]).

We define the map $j:A\to A$ as follows $j:(n,k)\to(n+m,k)$ and consequently $j^s:(n,k)\to(n+sm,k)$ and m is chosen in such a way that $\frac{1}{\mu^m}$ to compensate ω^{m+2} (cf. Appendix C).

In this manner we obtain a uniform estimate for the contractive constant smaller than 1 because we choocs $\omega<\mu$. Besides it is easy to see that

$$K^{(m,k)}K^{(m+1,k)}...K^{(n+s(m+1),k)}/K^{(m,k)}K^{(n+m,k)}...K^{(n+sm,k)}=$$
$$=K^{(n+(s+1)m,k)}\le K<1.$$

Therefore the operator B is contractive in the sense of [23]. It remains to show that the uniform space $M_r\times M_\phi$ is j-bounded (cf. [23]). Indeed,

$$\left|\frac{d^{n+sm}r(t)}{dt^{n+sm}}-\frac{dt^{n+sm}\bar{r}(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}\le$$
$$\le\left|\frac{d^{n+sm}r(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}+\left|\frac{d^{n+sm}\bar{r}(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}\le R_0\Rightarrow$$

$d_{(n+sm,k)}(r,\bar{r})\le R_0\ (s=0,1,2,...)$ for every k and

$$\left|\frac{d^{n+sm}\phi(t)}{dt^{n+sm}}-\frac{d^{n+sm}\bar{\phi}(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}\le$$
$$\le\left|\frac{d^{n+sm}\phi(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}+\left|\frac{d^{n+sm}\bar{\phi}(t)}{dt^{n+sm}}\right|e^{-\mu(t-kT)}\le\Phi_0$$

$(s=0,1,2,...)$ for every k.

Applying fixed point theorem from Chapter 2 from [23] we conclude that the operator B has a unique fixed point which is a solution of (6).

Theorem 1 is thus proved.

Remark 3. If we want to prove only existence without uniqueness, we suppose that $d_{(k,n+sm)}((r_0,\phi_0),B_{\phi,k}(r_0,\phi_0))\le Q(r_0,\phi_0,m)<\infty(s=0,1,2,\ldots)$ for some $(r_0,\phi_0)\in M_r\times M_\phi$.

VI. NUMERICAL EXAMPLE

For the first Bohr orbit we have $\rho_0=5{,}3.10^{-11}m$, while for the second one $\rho_1=2^2.5{,}3.10^{-11}m=21{,}2.10^{-11}m$ (cf. [24]). In order to describe the transition, we use (9), namely $\rho_0+(k_1-k_0)T=\rho_1$ and obtain

$$5{,}3.10^{-11}+(k_1-k_0)T=4.5{,}3.10^{-11}\Rightarrow k_1-k_0\cong 16.10^{-11}/T\,.$$

This estimate shows that the particle (electron) must perform $16.10^{-11}/T$ rounds to pass from the first stationary state to the second one.

We show that all energies of stationary states [11] can be included in the our estimate of the energy obtained in [12].

Indeed, following [24] we have $W_l=-\frac{1}{\varepsilon_0^2}\frac{e_2^4m_2}{8\hbar^2}\frac{1}{l^2}$ $(l=1,2,...)$. On the other hand, we have the following equation for the energy of the moving electron from [25]:

$$\frac{m_2c^3\langle\vec{u}^{(2)},\vec{u}^{(2)}\rangle}{\Delta_2^{\ 3}}=e_1e_2c\left\{\Delta_{21}^{\ 2}\frac{\langle\vec{u}^{(2)},\vec{\xi}^{(21)}\rangle-\langle\vec{u}^{(2)},\vec{u}^{(1)}\rangle\tau_{21}}{\left(c^2\tau_{21}-\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\rangle\right)^3}+\right.$$

$$+\frac{D_{21}}{\Delta_{21}^2}\frac{\left[\Delta_{21}^2\left\langle\vec{\xi}^{(21)},\dot{\vec{u}}^{(1)}\right\rangle+(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21})\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\right](\left\langle\vec{u}^{(2)},\vec{\xi}^{(21)}\right\rangle-\left\langle\vec{u}^{(2)},\vec{u}^{(1)}\right\rangle\tau_{21})}{(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21})^3}+$$

$$+\frac{D_{21}}{\Delta_{21}^2}\frac{\left[\left\langle\vec{u}^{(1)},\dot{\vec{u}}^{(1)}\right\rangle\left(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-\left\langle\vec{u}^{(1)},\vec{u}^{(1)}\right\rangle\tau_{21}\right)-\Delta_{21}^2\left\langle\vec{u}^{(2)},\dot{\vec{u}}^{(1)}\right\rangle\tau_{21}\right]}{\left(\left\langle\vec{\xi}^{(21)},\vec{u}^{(1)}\right\rangle-c^2\tau_{21}\right)^2}\Bigg\}-\frac{e_2^2}{m_2c^2}\frac{\left\langle\vec{u}^{(2)},\ddot{\vec{u}}^{(2)}\right\rangle}{\Delta_2{}^3}.$$

For the Kepler problem $\vec{x}^{(1)}=\vec{0}$ in view of $\Delta_{21}=c$, $\dfrac{m_2c^3\left\langle\vec{u}^{(2)},\vec{u}^{(2)}\right\rangle}{\Delta_2{}^3}=\dfrac{dE_{kin}^{(2)}}{dt}$ and $\tau_{21}=\dfrac{1}{c}\sqrt{\left\langle\vec{\xi}^{(21)},\vec{\xi}^{(21)}\right\rangle}=\dfrac{1}{c}\sqrt{\left\langle\vec{x}^{(2)},\vec{x}^{(2)}\right\rangle}$

we obtain $\dfrac{dE_{kin}^{(2)}}{dt}=-e_2{}^2\left(\dfrac{\left\langle\vec{u}^{(2)},\vec{x}^{(2)}\right\rangle}{\sqrt{\left\langle\vec{x}^{(2)},\vec{x}^{(2)}\right\rangle}^3}+\dfrac{1}{m_2}\dfrac{\left\langle\vec{u}^{(2)},\ddot{\vec{u}}^{(2)}\right\rangle}{c^5}\right)$.

In polar coordinates we have

$$\frac{dE_{kin}^{(2)}}{dt}=-e_2{}^2\left(\frac{\dot{\rho}}{\rho^2}+\frac{\ddot{\rho}\dot{\rho}+\ddot{\varphi}\rho^2\dot{\varphi}}{m_2c^5}\right)=-e_2{}^2\left(\frac{r}{\rho^2}+\frac{r\ddot{r}+\rho^2\phi\ddot{\phi}}{m_2c^5}\right).$$

We have to compare

$$E_{kin}^{(2)}(t)=-e_2{}^2\int_0^t\left(\frac{r(s)}{\rho^2(s)}+\frac{r(s)\ddot{r}(s)+\rho^2(s)\phi(s)\ddot{\phi}(s)}{m_2c^5}\right)ds$$

with $W_l=-\dfrac{1}{\varepsilon_0{}^2}\dfrac{e_2{}^4m_2}{8\hbar^2}\dfrac{1}{l^2}$, $(l=1,2,...)$.

Since

$$\left|E_{kin}^{(2)}(t)\right|\le e_2{}^2\int_0^t\left(\frac{R_0e^{\mu s}}{\hat{\rho}^2}+\omega^2\,\frac{R_0{}^2e^{2\mu s}+\bar{c}\breve{\rho}\Phi_0^2e^{2\mu s}}{m_2c^5}\right)ds\le$$

$$\le e_2{}^2\int_0^t\left(\frac{\bar{c}}{\hat{\rho}^2}+\omega^2\,\frac{\bar{c}^2+\bar{c}^3}{m_2c^5}\right)ds\le$$

$$\le e_2{}^2\left(\frac{\bar{c}}{\hat{\rho}^2}+\frac{2\bar{c}^3\omega^2}{m_2c^5}\right)\frac{e^{\mu T}}{\mu}\le e_2{}^2\bar{c}\left(\frac{1}{\hat{\rho}^2}+\frac{2\omega^2}{m_2c^2}\right)\frac{e^{\mu T}}{\mu}$$

we have to show that $\dfrac{1}{\varepsilon_0{}^2}\dfrac{e_2{}^4m_2}{8\hbar^2}\le e_2{}^2\bar{c}\left(\dfrac{1}{\hat{\rho}^2}+\dfrac{2\omega^2}{m_2c^2}\right)\dfrac{e^{\mu T}}{\mu}$.

Indeed, for $\omega=4{,}15.10^{16}$ (cf. [5]), $\rho_0=0{,}53.10^{-10}m$, $\mu=10^{16}, T=10^{-15}\Rightarrow\mu T=10\Rightarrow e^{10}\approx 2{,}2.10^4$, $1+(e^{\mu T}-1)/\mu=1+2{,}2.10^4.10^{-16}\approx 1$, $\bar{c}=2{,}2.10^6$.

If we choose, for instance, $R_0=1$ we obtain

$$\hat{\rho}=\rho_0-\frac{R_0(e^{\mu T}-1)}{\mu}=5{,}3.10^{-11}-R_0\,\frac{2{,}2.10^4}{10^{16}}=$$

$$=5{,}3.10^{-11}-0{,}22.R_0.10^{-11}=5{,}08.10^{-11}$$

and then

$$\frac{1}{(8{,}86.10^{-12})^2}\frac{(1{,}64.10^{-19})^2.9{,}11.10^{-31}}{8(6{,}62.10^{-34})^2}\le$$

$$\le\left(\frac{1}{(5{,}08.10^{-11})^2}+\frac{2(4{,}15.10^{16})^2}{9{,}11.10^{-31}.9.10^{16}}\right)\frac{2{,}2.10^6.2{,}2.10^4}{10^{16}}$$

or $8{,}5.10^{19}\le\left(3{,}8.10^{14}+4{,}2.10^{40}\right).4{,}84$.

Consequently, our interval of the energy includes all discrete values of the stationary energy.

Relations of the above results are possible not only with the classical areas but also with the recent model obtained in [26].

VII. CONCLUSIONS

In this paper, we showed the existence of trajectories in the transition of a particle (electron) from a stationary state to a state with a larger radius. The approach is also applicable to the N-body problem. In this way, we propose a deterministic alternative to the stochastic approach introduced by M. Planck, who actually transfers methods from thermodynamics to electrodynamics.

The result obtained here, extends the results of our previous papers, where we found the conditions for existence and uniqueness of periodic orbit of the two-body problem in classical electrodynamics. In future papers, we will show the existence of periodic orbits in the space of spherical coordinates, as it turns out that the physical interpretation is clearer. In addition, the transition from an orbit with a larger radius to one with a smaller one is also possible, which we will also consider in the next papers.

APPENDIX A

Lemma 2. The sets M_r and M_ϕ are closed.

Proof: 1) Let us choose a sequence $\{r_{(l)}\}_{l=1}^{\infty}$, $r_{(l)}\in M_r$ $r_{(l)}\underset{l\to\infty}{\to}r$. Therefore

$$r_1-\int_{kT}^{(k+1)T}r(s)\,ds-r_1-\int_{kT}^{(k+1)T}r_{(l)}(s)ds\le$$

$$\le\left|\int_{kT}^{(k+1)T}(r(s)-r_{(l)}(s))\,ds\right|\le d_{(m,k)}(r,r_{(l)})\chi^m e^{\mu T}\le\varepsilon\chi^m e^{\mu T}.$$

Consequently, $\displaystyle\int_{kT}^{(k+1)T}r_{(l)}(s)ds=0\Rightarrow\int_{kT}^{(k+1)T}r(s)ds=0$ and $\displaystyle\int_{kT}^{(k+1)T}r_l(s)ds=r_1T\;\Rightarrow\int_{kT}^{(k+1)T}r(s)ds=r_1T$.

For $\displaystyle\int_{k_0T}^{(k_0+1)T}r(s)ds$ we have

$$\int_{k_0T}^{(k_0+1)T} r(s)ds = \int_{k_0T}^{k_0T+\varepsilon} r(s)ds + \int_{k_0T+\varepsilon}^{(k_0+1)T} r(s)ds \text{ and}$$

$$\int_{k_0T}^{(k_0+1)T} r(s)ds = \int_{k_0T}^{k_0T+\varepsilon} r(s)ds + \int_{k_0T+\varepsilon}^{(k_0+1)T} r(s)ds \le$$

$$\le \int_{k_0T}^{k_0T+\varepsilon} r_{(l)}(s)ds + \int_{k_0T+\varepsilon}^{(k_0+1)T} r_{(l)}(s)ds = \int_{k_0T}^{(k_0+1)T} r_{(l)}(s)ds \le r_1 T.$$

The similar reasining can be repeated for $\int_{(k_1-1)T}^{k_1T} r_{(l)}(s)ds$.

2) $$\int_{kT}^{(k+1)T} \phi(s)\,ds - \int_{kT}^{(k+1)T} \phi_{(l)}(s)ds \le \left| \int_{kT}^{(k+1)T} (\phi(s) - \phi_{(l)}(s))\,ds \right| \le$$

$$\le d_{(m,k)}(\phi, \phi_{(l)})\chi^m e^{\mu T} \le \varepsilon \chi^m e^{\mu T}.$$

Then $\int_{kT}^{(k+1)T} \phi_{(l)}(s)ds = \phi_0 T \Rightarrow \int_{kT}^{(k+1)T} \phi(s)ds = \phi_0 T$ which proves Lemma 2.

APPENDIX B

It is easy to see that if we choose $\left| r^{(m)}(kT) \right| = \frac{1}{\mu} \approx 0$, $(m = 1,...,n)$, then for $k = k_0,...,k_1$ denoting $\chi = \omega/\mu < 1$ for $k = k_0,...,k_1$ we obtain

1) $$|r(t)| = \left| r(kT) + \int_{kT}^{t} \dot{r}(s)ds \right| =$$

$$= \left| r_1 + \int_{kT}^{t} \dot{r}(kT)dt_1 + ... + \int_{kT}^{t}\int_{kT}^{t_1} ... \int_{kT}^{t_{n-1}} r^{(n)}(t_n)dt_n...dt_2dt_1 \right| \le$$

$$\le \left(r_1 + \frac{|\dot{r}(kT)|}{\mu} + ... + \frac{\omega^n}{\mu^n} R_0 \right) e^{\mu(t-kT)} =$$

$$= \left(r_1 + \frac{\omega}{\mu^2} R_0 + ... + \frac{\omega^n}{\mu^{n+1}} R_0 \right) e^{\mu(t-kT)} \approx \left(r_1 + \frac{1}{\mu}\chi^n R_0 \right) e^{\mu T};$$

2) $$|\dot{r}(t)| = \left| \dot{r}(kT) + \int_{kT}^{t} \ddot{r}(s)ds \right| =$$

$$= \left| \dot{r}(kT) + ... + \int_{kT}^{t}\int_{kT}^{t_1} ... \int_{kT}^{t_{n-1}} r^{(n)}(t_n)dt_n...dt_2dt_1 \right| \le \frac{\chi^n}{\mu} R_0 e^{\mu T}$$

and so on.

3) $$|\phi(t)| = \left| \phi(kT) + \int_{kT}^{t} \dot{\phi}(s)ds \right| =$$

$$= \left| \phi_0 + \int_{kT}^{t} \dot{\phi}_0 dt_1 + ... + \int_{kT}^{t}\int_{kT}^{t_1} ... \int_{kT}^{t_{n-1}} \phi^{(n)}(t_n)dt_n...dt_2dt_1 \right| \le$$

$$\le \left(\phi_0 + \frac{|\dot{\phi}_0|}{\mu} + ... + \frac{\omega^n}{\mu^n}\Phi_0 \right) e^{\mu(t-kT)} =$$

$$= \left(\phi_0 + \frac{1}{\mu^2} + ... + \frac{\omega^n}{\mu^n}\Phi_0 \right) e^{\mu(t-kT)} \approx \left(\phi_0 + \chi^n \Phi_0 \right) e^{\mu T};$$

4) $$|\dot{\phi}(t)| = \left| \dot{\phi}(kT) + \int_{kT}^{t} \ddot{\phi}(s)ds \right| =$$

$$\left| \dot{\phi}_0 + \int_{kT}^{t} \ddot{\phi}_0 dt_1 + ... + \int_{kT}^{t}\int_{kT}^{t_1} ... \int_{kT}^{t_{n-1}} \phi^{(n+1)}(t_n)dt_n...dt_2dt_1 \right| \le \omega\chi^n \Phi_0 e^{\mu T};$$

5) $$|r(t) - \bar{r}(t)| = \left| \int_{kT}^{t} (\dot{r}(s) - \dot{\bar{r}}(s))ds \right| \le$$

$$\le \int_{kT}^{t} \left| \dot{r}(s) - \dot{\bar{r}}(s) \right| e^{-\mu(s-kT)} e^{\mu(s-kT)} ds \le d_{(1,k)}(r, \bar{r}) \frac{e^{\mu(t-kT)}}{\mu}$$

$$\Rightarrow d_{(0,k)}(r, \bar{r}) \le \frac{d_{(1,k)}(\dot{r}, \dot{\bar{r}})}{\mu};$$

6) $$|\phi(t) - \bar{\phi}(t)| = \left| \int_{kT}^{t} (\dot{\phi}(s) - \dot{\bar{\phi}}(s))ds \right| \le$$

$$\le \int_{kT}^{t} \left| \dot{\phi}(s) - \dot{\bar{\phi}}(s) \right| e^{-\mu(s-kT)} e^{\mu(s-kT)} ds \le d_1^{\,k}(\dot{\phi}, \dot{\bar{\phi}}) \frac{e^{\mu(t-kT)}}{\mu} \Rightarrow$$

$$\Rightarrow d_{(0,k)}(\phi, \bar{\phi}) \le \frac{d_{(1,k)}(\dot{\phi}, \dot{\bar{\phi}})}{\mu}.$$

APPENDIX C

Estimates for partial derivatives of F_r and F_ϕ

In view of system (6) and using the inequalities $r^2 + \rho^2\phi^2 \le R_0^{\,2} + (\breve{\rho}_1)^2 \Phi_0^{\,2} e^{2\mu T} \le \bar{c}^2$ $(R_0 \le \bar{c})$ we get

$$\frac{\partial F_r}{\partial \rho} = \phi^2 - \frac{e_1 e_2}{m_2} \frac{c^2 - r^2}{c^2} \frac{2c^2 - 2r^2 - \rho^2\phi^2}{\rho^3 \Delta_2} -$$

$$-\frac{e_2^2}{m_2 c^2} \left(2\frac{r\phi\rho\ddot{\phi} - \phi^2\rho\ddot{r}}{\Delta_2^{\,3}} + 3\frac{\ddot{\phi}\rho^3\phi^3 r + (c^2 - \rho^2\phi^2)\rho\phi^2\ddot{r}}{\Delta_2^{\,5}} \right);$$

$$\frac{\partial F_r}{\partial r} = -\frac{e_1 e_2}{m_2 c^2 \rho^2} r \frac{3c^2 - 3r^2 - 2\rho^2\phi^2}{\sqrt{c^2 - r^2 - \rho^2\phi^2}} -$$

$$\frac{e_2^2}{m_2 c^2} \left(\frac{\ddot{\phi}\rho^2\phi}{\Delta_2^{\,3}} + 3r\frac{\ddot{\phi}\rho^2 r\phi + (c^2 - \rho^2\phi^2)\ddot{r}}{\Delta_2^{\,5}} \right);$$

$$\frac{\partial F_r}{\partial \phi}=2\rho\phi-\frac{e_1e_2}{m_2c^2}\frac{\left(c^2-r^2\right)\phi}{\sqrt{c^2-r^2-\rho^2\phi^2}}-$$
$$\frac{e_2^2}{m_2c^2}\left(\frac{\ddot{\phi}\rho^2 r-2\rho^2\phi)\ddot{r}}{\Delta_2{}^3}-3\rho^2\phi\frac{\ddot{\phi}\rho^2 r\phi+(c^2-\rho^2\phi^2)\ddot{r}}{\Delta_2{}^5}\right);$$

$$\frac{\partial F_r}{\partial \rho}=\phi^2-\frac{e_1e_2}{m_2}\frac{c^2-r^2}{c^2}\frac{2c^2-2r^2-\rho^2\phi^2}{\rho^3\Delta_2}-$$
$$-\frac{e_2^2}{m_2c^2}\left(2\frac{r\phi\rho\ddot{\phi}-\phi^2\rho\ddot{r}}{\Delta_2{}^3}+3\frac{\ddot{\phi}\rho^3\phi^3 r+(c^2-\rho^2\phi^2)\rho\phi^2\ddot{r}}{\Delta_2{}^5}\right);$$

$$\frac{\partial F_r}{\partial r}=-\frac{e_1e_2}{m_2c^2\rho^2}r\frac{3c^2-3r^2-2\rho^2\phi^2}{\sqrt{c^2-r^2-\rho^2\phi^2}}-$$
$$\frac{e_2^2}{m_2c^2}\left(\frac{\ddot{\phi}\rho^2\phi}{\Delta_2{}^3}+3r\frac{\ddot{\phi}\rho^2 r\phi+(c^2-\rho^2\phi^2)\ddot{r}}{\Delta_2{}^5}\right);$$

$$\frac{\partial F_r}{\partial \phi}=2\rho\phi-\frac{e_1e_2}{m_2c^2}\frac{\left(c^2-r^2\right)\phi}{\sqrt{c^2-r^2-\rho^2\phi^2}}-$$
$$\frac{e_2^2}{m_2c^2}\left(\frac{\ddot{\phi}\rho^2 r-2\rho^2\phi)\ddot{r}}{\Delta_2{}^3}-3\rho^2\phi\frac{\ddot{\phi}\rho^2 r\phi+(c^2-\rho^2\phi^2)\ddot{r}}{\Delta_2{}^5}\right);$$

$$\left|\frac{\partial F_r}{\partial \rho}\right|\le\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)^2+$$
$$+\frac{|e_1e_2|}{m_2}\left(\frac{2c}{\widehat{\rho}^3(1-\beta^2)^{1/2}}+2\frac{|r\ddot{\phi}+\phi\ddot{r}|}{c^4(1-\beta^2)^{3/2}}+3\frac{|\ddot{\phi}r+\phi\ddot{r}|}{c^4(1-\beta^2)^{5/2}}\right)\le$$

$$\le\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)^2+\frac{5|e_1e_2|}{m_2(1-\beta^2)^{5/2}}\left(\frac{c}{\widehat{\rho}^3}+\frac{\omega^2\chi^n(2+\chi^n)}{\breve{\rho}c^2}\right);$$

$$\left|\frac{\partial F_r}{\partial r}\right|\le\frac{|e_1e_2|}{m_2}\left(\frac{3r}{c\rho^2(1-\beta^2)^{1/2}}+\frac{c^2\ddot{\phi}\rho+3\ddot{\phi}\rho r^2+3cr\ddot{r}}{c(1-\beta^2)^{5/2}}\right)\le$$
$$\le\frac{|e_1e_2|}{m_2}\left(\frac{3}{\widehat{\rho}^2(1-\beta^2)^{1/2}}+\frac{c\breve{\rho}\phi_0+7c^2\omega^2\chi^n}{(1-\beta^2)^{5/2}}\right);$$

$$\left|\frac{\partial F_r}{\partial \phi}\right|\le 2c+\frac{|e_1e_2|}{m_2c}\left[\frac{\phi}{(1-\beta^2)^{1/2}}+\frac{4\ddot{\phi}r\breve{\rho}+2\phi\ddot{r}\breve{\rho}+3c\ddot{r}}{c^4(1-\beta^2)^{5/2}}\breve{\rho}\right]\le$$
$$\le 2c+\frac{|e_1e_2|}{m_2c}\left[\frac{\phi_0+\chi^n\Phi_0e^{\mu T}}{(1-\beta^2)^{1/2}}+\frac{9+2\chi^ne^{\mu T}}{c^2(1-\beta^2)^{5/2}}\omega^2\chi^n\breve{\rho}\right];$$

$$\left|\frac{\partial F_\phi}{\partial \rho}\right|\le\frac{2\bar{c}^2\chi^n}{\widehat{\rho}^2\breve{\rho}}+$$
$$+\frac{|e_1e_2|\chi^n}{m_2}\left(\frac{2c^2\bar{c}}{\widehat{\rho}^3\breve{\rho}(1-\beta^2)^{1/2}}+4\omega^2\Phi_0\frac{1+\chi^ne^{\mu T}+e^{\mu T}}{c^3(1-\beta^2)^{5/2}}\right);$$

$$\left|\frac{\partial F_\phi}{\partial r}\right|\le\frac{2\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)}{\widehat{\rho}}+$$
$$+\frac{|e_1e_2|}{m_2}\left(\frac{2c\bar{c}\left(1+\chi^n\right)}{\widehat{\rho}^2\breve{\rho}(1-\beta^2)^{1/2}}+\frac{3\omega^2\chi^n+3\omega^2\chi^n+3e^{\mu T}}{c^3(1-\beta^2)^{5/2}}\right);$$

$$\left|\frac{\partial F_\phi}{\partial \phi}\right|\le\frac{2\bar{c}}{\widehat{\rho}}+$$
$$+\frac{|e_1e_2|}{m_2}\left(\frac{c\bar{c}}{\widehat{\rho}^2}+\frac{\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)^2}{(1-\beta^2)^{1/2}}+\frac{\omega^2\chi^n(4\breve{\rho}+3)}{c^3(1-\beta^2)^{5/2}}\right).$$

Estimates for time derivatives of F_r and F_ϕ

$$\left|\dot{F}_r\right|\le\bar{c}\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)^2+\frac{2c\bar{c}\omega\chi^n}{\breve{\rho}}+$$
$$\frac{|e_1e_2|}{m_2}\left(\frac{2\omega\chi^n}{\widehat{\rho}^2}+\frac{2\chi^n+\Phi_0e^{\mu T}+\chi^n\Phi_0e^{\mu T}}{\widehat{\rho}^2\left(1-\beta^2\right)^{1/2}}+\right.$$
$$+\frac{2\bar{c}e^{\mu T}}{\widehat{\rho}^3}+\frac{3\omega^2\chi^n}{c^2\left(1-\beta^2\right)^{3/2}}+$$
$$\left.+\frac{6\omega^2\chi^n\left(2\omega\chi^n+\phi_0e^{\mu T}+\chi^n\Phi_0e^{\mu T}\right)}{c^2\left(1-\beta^2\right)^{5/2}}\right);$$

$$\left|\dot{F}_\phi\right|\le 2\omega\chi^n\bar{c}\Phi_0\frac{1+\chi^ne^{\mu T}+e^{\mu T}}{\widehat{\rho}}+\frac{2\bar{c}^2\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)}{\widehat{\rho}^2}+$$
$$+\frac{|e_1e_2|}{m_2}\left(4\omega\chi^nc\bar{c}\Phi_0\frac{1+\chi^ne^{\mu T}+e^{\mu T}}{\breve{\rho}^2}+\right.$$
$$+\frac{2c^3\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)}{\widehat{\rho}^3}+\frac{\bar{c}\left(\phi_0+\chi^n\Phi_0e^{\mu T}\right)^2}{\widehat{\rho}^2}+$$
$$+\frac{5\bar{c}+2\breve{\rho}\chi^n\omega e^{\mu T}+\breve{\rho}+\omega}{c^2\left(1-\beta^2\right)^{3/2}}\omega^2\chi^n+$$
$$\left.3\omega^2\chi^n\frac{(\phi_0\breve{\rho}+\breve{\rho})\left(\omega\chi^n\bar{c}+\phi_0e^{\mu T}+\chi^n\Phi_0e^{\mu T}+\omega\chi^ne^{\mu T}\right)}{c\left(1-\beta^2\right)^{5/2}}\right).$$

The higher derivatives can be estimated in the similar way.

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