Extended Poisson's Theory for Analysis of Bending of a Simply Supported Square Plate

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Abstract: *Nature of solutions from different methods of analysis is examined with reference to exact solution of text book problem of bending of a simply supported square plate under bi-sinusoidal load. Methods of analysis based on stationary property of total potential correspond to plate element equilibrium equations. In these methods, vertical displacement is a domain variable like in 3-D equations of equilibrium in terms of displacements. Aim of the present work is to show that these methods deal with solution of associated torsion problem instead of bending problem. Solution of bending problem is only through methods based on vertical displacement as a face variable.*

Key words: *Plates, Bending, Isotropy, Elasticity*

1 INTRODUCTION

Kirchhoff's theory [1] of bending of plates is well-known classical theory and widely used even to-day by practicing engineers due to its simplicity in obtaining design information. In this theory, vertical deflection $w_0(x, y)$ is governed by a fourth order differential equation along with two boundary conditions instead of three conditions required in a 3-D problem. This lacuna in the theory is the well-known Poisson-Kirchhoff boundary conditions paradox. Resolution of this paradox requires formulation of a sixth order theory. It took more than nine decades to provide such a theory through Reissner's pioneering work [2]. Reissner's theory is a stress based theory but stress resultants are derived in terms of a Lagrange multiplier recognised later as an average vertical displacement $w_0(x, y)$. Soon after, displacement based sixth order theory was followed. This theory is based on Hencky's theory [3] and known as First Order Shear Deformation Theory abbreviated as FSDT. It is widely used with presumption of better range of applicability than Kirchhoff's theory and offers simplicity in the use of finite element methods. In FSDT, however, zero transverse shear conditions along faces of the plate are not satisfied and the influence of their thickness-wise parabolic distributions in the interior of the plate is included through shear correction factor. It is derived through distribution correction factor so as to derive corresponding factors in higher order theories [4]. Several sixth order shear deformation theories to avoid shear correction factor and higher order shear deformation and other theories are reported in the Literature; for example, see [5] and [6]. Most of these theories are generally based on plate element equilibrium equations (PEEES) in which $w_0(x, y)$ is used as a domain variable. St.Venant's torsion problem in which normal strains are zero is often used to justify sixth order theories. It is not proper for this purpose since $w_0(x, y)$ is a priory known due to assumptions in St.Venant's torsion problem. Associated torsion problem in bending is, however, different from bending problem. Even the exact solution of 3-D equations in terms of displacements corresponds to the torsion problem mainly due to vertical displacement being a domain variable

A new concept of using $w_0(x, y)$ as face variable was introduced in recent article [6, 7]. It is an essential requirement for the analysis of bending of plates. New theory dealing with parabolic distribution of reactive transverse shear stresses is designated as "Poisson's theory of plates in bending" and its modification, in which assumed transverse shear stresses are independent of thickness co-ordinate, is designated as "Extended Poisson's theory". In these theories, primary transverse stresses are independent of material constants thereby independent if elastic deformations. It is complimentary to the fact that in-plane stresses are independent of material constants in the analysis of extension problems through Airy's stress function.

Prescribed upper and bottom face conditions along with edge conditions can be modified such that even functions $f_{2n}(z)$ and odd functions $f_{2n+1}(z)$ in the z-distribution of in-plane displacements are for analysis of extension and bending problems, respectively. Correspondingly, vertical displacement $w(x, y, z)$ is odd and even in the extension and bending problems, respectively, due to transverse shear strain-displacement relations. As such, displacements in the bending problems are assumed as, with $f_k(z)$ in [6, 7],

$$[u, v, w] = f_k(z)[u_{2k+1}, v_{2k+1}, w_{2k}]$$

II BENDING OF SIMPLY SUPPORTED SQUARE PLATE

A square plate bounded within $0 \leq X, Y \leq a$, $+h \geq Z \geq - h$ with reference to Cartesian coordinate system (X, Y, Z) is considered. Material of the plate is homogeneous and isotropic with elastic constants E (Young's modulus), ν (Poisson's ratio) and G (Shear modulus) that are related to one other by $E = 2(1+\nu) G$. For convenience, coordinates X, Y, Z and displacements (U, V, W) in non-dimensional form $x= X/a$, $y=Y/a$, $z=Z/h$, $(u, v, w) = (U, V, W)/h$ and half-thickness ratio $\alpha = (h/a)$ are used. With the above notation, equilibrium equations in stress components are:

$$\alpha(\sigma_{x,x}+ \tau_{xy,y}) + \tau_{xz,z} = 0,\ \alpha(\sigma_{y,y}+ \tau_{xy,x}) + \tau_{xz,z} = 0$$
$$\alpha(\tau_{xz,x}+ \tau_{yz,y}) + \sigma_{z,z} = 0$$

in which suffix after ',' denotes partial derivative operator. The plate is subjected to asymmetric load $\sigma_z = \pm (q_0/2) \sin(\pi x) \sin(\pi y)$ and zero shear stresses along $z = \pm 1$ faces. In a simply supported plate, conditions along x = constant edges (with analogous conditions along y = constant edges) are $\sigma_x = 0$, $v = 0$, $w = 0$.

Present work is exclusively concerned with displacement based models. Here, stress components are expressed in displacements, via, strain-displacement and stress-strain constitutive relations. These relations within the classical small deformation theory of elasticity are:

Strain-displacement relations: $[\varepsilon_x, \varepsilon_y, \varepsilon_z] = [\alpha u_{,x}, \alpha v_{,y}, w_{,z}]$, $[\gamma_{xy}, \gamma_{xz}, \gamma_{yz}] = [\alpha u_{,y}+ \alpha v_{,x}, u_{,z}+ \alpha w_{,x}, v_{,z}+ \alpha w_{,y}]$
Constitutive relations: $E\varepsilon_x= \sigma_x- \nu (\sigma_y+ \sigma_z)$, $E\varepsilon_y= \sigma_y- \nu (\sigma_x+ \sigma_z)$, $E\varepsilon_z= \sigma_z- \nu (\sigma_x+ \sigma_y)$, and $[\tau_{xy}, \tau_{xz}, \tau_{yz}] = G [\gamma_{xy}, \gamma_{xz}, \gamma_{yz}]$

III METHODS OF ANALYSIS

The purpose of the present work is to assess the nature of solutions from different methods of analysis. Here, methods of analysis are broadly classified into the following three groups: (i) Exact solutions in terms of displacements, (ii) Sequence of 2-D problems based on plate element equilibrium equations, and (iii) Sequence of 2-D problems based on infinitesimal element equilibrium equations

A. Exact Solutions: In the case of a square plate, exact solutions for $w_0(x, y)$ used as both domain and face variable are reported in [6].

B. Theories with $w_0(x, y)$ as domain variable

Kirchhoff's theory, Reissner's theory, and FSDT are based on PEEES. These equations are in terms of stress resultants and/or average displacements using calculus of variations though actual 3-D equations are independent of these quantities. Unfortunately, they are approximate equations governing associated torsion problems instead of bending problems. Kirchhoff's theory due to presence of transverse shear stress resultants may be considered as 0^{th} order shear deformation theory.

In FSDT, transverse shear stress components (τ_{xz2}, τ_{yz2}) are expressed earlier [4] in terms of (τ_{xzo}, τ_{yz0}) through distribution correction factor $\beta_0 = (5/2)$ obtained from

$$[½(1- z^2)\, \tau^* - \beta_0 \tau](1- z^2)\, dz = 0$$

integrated from bottom to top faces of the plate giving $\tau^* = \beta_0 \tau$ so that

$$(\tau^*_{xz}, \tau^*_{yz}) = (5/4)\, (1- z^2)\, G(u_1 + \alpha w_{0,x},\ v_1 + \alpha w_{0,y})$$

resulting in shear energy correction factor $k^2 = 5/6$. With $w_0 = c_0 \sin(\pi x) \sin(\pi y)$, the expression for c_0 in the case of a square plate is

$$c_0 = q\, [1 + \frac{6}{5(1-\nu)} (2\, \alpha^2\, \pi^2)]/[4\, \alpha^4 \pi^4 D']$$

IV HIGHER ORDER THEORIES

In 2-D plate theories with $w_0(x, y)$ as domain variable, one cannot avoid linear thickness-wise distributions of in-plane displacements. Several two term representations of displacements are used primarily intended to avoid the use of shear correction factor in FSDT. These displacements are of the form

$$w = w_0(x, y) + f_{,z}(z)\, w_2(x, y)$$
$$[u, v] = z\, [u_1, v_1] + f(z)\, [u_3, v_3]$$

Various theories are due to the choice of f(z) and six 2-D variables in the above equation. Lower order theories based on a priory satisfying zero face shear conditions are

$$w = w_0(x, y) + f_{,z}(z)\, w_2(x, y)$$
$$[u, v] = - z\, \alpha\, [w_{0,x}, w_{0,y}] + f(z)\, [u_3, v_3]$$

with assumed f(z) in different forms listed in [6] are

(i)$f_3(z)$, (ii) $\frac{2}{\pi} \sin \frac{\pi}{2} z$, (iii) $\sinh \frac{z}{2} - \frac{z}{2} \cosh \frac{1}{2}$,

(iv) $z\exp(-\frac{1}{2}z^2)$,(v) $(7z - 4z^3 + z^5)/8$,(vi) $(\tan^{-1}z - z)$,

(vii) $\tan^{-1}(\sin\frac{\pi}{2}z)$, (viii) $z\,[\sec(r\frac{z}{2}) - \frac{\sec\frac{r}{2}}{1+\frac{r}{2}\tan\frac{r}{2}}]$

Apart from FSDT, one could consider several sixth order theories using f(z) like in the above list in conjunction with displacements

$$w = w_0(x, y) \quad ; \quad [u, v] = -z\,\alpha\,[w_{0,x}, w_{0,y}] + f(z)\,\alpha\,[\psi_{,x}, \psi_{,y}]$$

A. Higher order theories based on FSDT

Adapting the concept of shear correction factor in FSDT, higher order transverse shear terms with $f_{2k+2}(z)$ may be expressed in terms of preceding shear terms with $f_{2k}(z)$ through distribution correction factors β_{2k} so that

$$[\tau_{xz}, \tau_{yz}] = \Sigma f_{2k}\,[\tau_{xz}, \tau_{yz}]_{2k}$$
(using Strain-Displacement relations)
$$[\tau_{xz}, \tau_{yz}] = \Sigma \beta_{2k} f_{2k+2}\,[\tau_{xz}, \tau_{yz}]_{2k}$$
(based on shear correction factors)

In such a case, solutions of plate element equations give shear strains $[u_1 + \alpha w_{0,x}\ ,\ v_1 + \alpha w_{0,y}]$ tending to [0, 0] in the limit $k \to \infty$ due to $[\tau_{xz}, \tau_{yz}]$ in the first set but not zero due to stresses in the second set. Obviously, shear energy due to β_{2k} does not belong to the physical problem.

In order to generate a converging sequence of 2-D problems with $w_0(x, y)$ as domain variable, it is more convenient to express z in Fourier series. Due to zero face shear conditions, it is expressed in the form

$$z = \sum A_k \sin kz \quad (k = 1, 2, 3, \ldots..)$$
$$A_k = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} z\sin kz\,dz = \frac{2}{\pi}\left(\frac{1}{k}\right)^2 [\sin k\frac{\pi}{2} - k\frac{\pi}{2}\cos k\frac{\pi}{2}]$$

so that z-distribution g(z) of reactive transverse stresses is given by $g(z) = -\sum A_k\frac{1}{k}\cos kz$. (Note that term by term differentiation is not valid in the series expansion of z)

Displacements are assumed in the form with k = 1, 2,…

$$w = w_0(x, y) - \Sigma\,[A_k\frac{1}{k}\cos kz]\,w_{2k}(x, y)$$
$$[u, v] = z\,[u, v]_1 + \Sigma\,[(\tfrac{1}{k})^2 \sin kz]\,[u, v]_{2k+1}$$
$$= \Sigma\,\{A_k\,[u, v]_1 + (\tfrac{1}{k})^2\,[u, v]_{2k+1}\}\sin kz$$

Transverse shear strains from strain-displacement relations and shear stresses are

$$[\gamma_{xz}, \gamma_{yz}] = [u, v]_1 + \alpha\,[w_{0,x}, w_{0,y}] + \Sigma\,\{[u, v]_{2k+1} - A_k\alpha[w_{2k,x}, w_{2k,y}]\}\,\frac{1}{k}\cos kz$$
$$[\tau_{xz}, \tau_{yz}] = G\,[\gamma_{xz}, \gamma_{yz}]$$

Here, it is convenient to consider equations of equilibrium in the form

$$[\tau_{xz}, \tau_{yz}] = -[\int \alpha(\sigma_{x,x} + \tau_{xy,y})\,dz\ ,\ \int \alpha(\sigma_{y,y} + \tau_{xy,x})\,dz]$$
$$\sigma_z = -\int \alpha(\tau_{xz,x} + \tau_{yz,y})\,dz$$

After some algebra, above equations of $[\tau_{xz}, \tau_{yz}]$ with obvious notation become

$$G\{u_1 + \alpha w_{0,x} + \Sigma\,[u_{2k+1} - A_k\alpha w_{2k,x}]\,\frac{1}{k}\cos kz\} = $$
$$= A(x, y) + \Sigma[A_k(\sigma_{x,x} + \tau_{xy,y})_1 + (\tfrac{1}{k})^2(\sigma_{x,x} + \tau_{xy,y})_{2k+1}]\,(\frac{\alpha}{k})\cos kz$$
$$G\{v_1 + \alpha w_{0,y} + \Sigma\,[v_{2k+1} - A_k\alpha w_{2k,y}]\,\frac{1}{k}\cos kz\} = $$
$$= B(x, y) + \Sigma[A_k(\sigma_{y,y} + \tau_{xy,x})_1 + (\tfrac{1}{k})^2(\sigma_{y,y} + \tau_{xy,x})_{2k+1}]\,(\frac{\alpha}{k})\cos kz$$

Above equations give $A(x, y) = G(u_1 + \alpha w_{0,x})$ and $B(x, y) = G(v_1 + \alpha w_{0,y})$. (Note that A(x, y) and B(x, y) are zero in FSDT but not the corresponding strains. Shear energy correction factor k^2 is through distribution correction factor expressed above in series form)

$$G[u_{2k+1} - A_k\alpha w_{2k,x}] =$$
$$\alpha\,[A_k(\sigma_{x,x} + \tau_{xy,y})_1 + (\tfrac{1}{k})^2(\sigma_{x,x} + \tau_{xy,y})_{2k+1}]$$
$$G[v_{2k+1} - A_k\alpha w_{2k,y}] =$$
$$\alpha\,[A_k(\sigma_{y,y} + \tau_{xy,x})_1 + (\tfrac{1}{k})^2(\sigma_{y,y} + \tau_{xy,x})_{2k+1}]$$

Zero face shear conditions give $[u_1, v_1] = -\alpha\,[w_{0,x}, w_{0,y}]$ so that

$$\sigma_z = -G\Sigma\,[e_{2k+1} - A_k\alpha^2\Delta w_{2k}]\,(\tfrac{1}{k})^2 \sin kz\}$$
$$= -\Sigma\,[A_k(\sigma_{x,xx} + 2\tau_{xy,xy} + \sigma_{y,yy})_1 +$$
$$+ (\tfrac{1}{k})^2(\sigma_{x,xx} + 2\tau_{xy,xy} + \sigma_{y,yy})_{2k+1}](\frac{\alpha}{k})^2 \sin kz$$

Due to Kirchhoff's displacements, σ_z confining to the first expression of the above equation is given by

$$\sigma_z = \Sigma\,[A_k(\frac{1}{k})^2 \sin kz\ \ (\alpha^4 E'\,\Delta\Delta w_0)$$

so that face load condition after some algebra gives Kirchhoff's equation E'D $\alpha^4\Delta\Delta w_0$ = q in which D = (2/3).

Above equation is to be solved with two edge conditions through the use of stress resultants from stationary property of total potential. As such, earlier statement that Kirchhoff's theory is, in a way, 0^{th} order shear deformation theory is justified.

From equations, we have

$$G[e_{2k+1} - A_k\alpha^2\Delta w_{2k}] = [A_k(\frac{1}{k})^2 q + (\frac{\alpha}{k})^2(\sigma_{x,xx} + 2\tau_{xy,xy} + \sigma_{y,yy})_{2k+1}]$$

Three variables w_{2k}, u_{2k+1}, v_{2k+1} are governed by the above equation and two in-plane equilibrium equations.

Above analysis clearly shows that the use of $w_0(x, y)$ as domain variable is not suitable for the analysis of bending problem (Hence, detailed algebra involved in the analysis is omitted). Moreover, use of stationary property of total potential leads to the solution of the associated torsion problem.

V THEORIES WITH W_0(X, Y) AS FACE VARIABLE

Exact solution of illustrative example of the present problem was obtained earlier [6] with assumed distributions in sinh and cosh functions in z for [u,v, τ_{xz}, τ_{yz}]. Due to zero face shear stresses, face variable $w_0(x, y)$ is evaluated by replacing [$u_{,z}$, $v_{,z}$] with $[u, v]_{z=1}$ in shear stress-strain relations so that

$$\alpha\, w_0(x, y) = -\int [u\, dx + v\, dy]_{z=1}$$

VI NUMERICAL RESULTS AND DISCUSSION

A typical thick plate of half-thickness ratio α = 1/6 with Poisson's ratio ν = 0.3 due to availability of solutions from various theories [8] is considered here for illustrative purpose.

With w_0 (x, y) as face variable, exact values of vertical displacement parameters [6] are

$$(E/2q_0)\, w(1/2, 1/2, 0) = 4.49$$
$$(E/2q_0)\, w(1/2, 1/2, 1) = 4.17$$

All shear deformation theories with two term representation of [u, v] give more or less same estimates around 3.48 to the neutral plane deflection $w_{0n}(x, y)$ from Ambartsumyan's theory [9].

One should note from Lewinski's article [8] that estimates to the vertical displacement parameter from various
12^{th} order theories are more or less equal to the exact neutral plane deflection 3.49. Numerical results from various lower order theories are given in the following Table.

Table 1: Face and Neutral Plane Displacements
$\hat{w}_{0f}$ = $(E/2q_0)$ w (1/2, 1/2, 1), $\hat{w}_{0n}$ = $(E/2q_0)$ w (1/2, 1/2, 0); α = 1/6, ν = 0.3

	1	2	3	4	5	6	7	8
$\hat{w}_{0f}$	4.12	4.17	2.27	2.54	3.72	3.69	4.07	4.07
$\hat{w}_{0n}$	3.49	4.49	2.27	2.54	3.72	3.41^a	4.36	4.46

First row: 1. *Exact: w(x, y, z) as domain variable,* 2. *Exact: w as face variable,* 3. *Kirchhoff's theory,* 4. *Poisson's theory,* 5. *Poisson's theory with one iteration,* 6. *FSDT (^{a} without k^2),* 7. *Extended Poisson's theory (without ε_{z1})* 8.*Extended Poisson's theory (with ε_{z1}). (Results in Poison's theories are based on presuming $\varphi_1 \equiv 0$)*

VII CONCLUSIONS

Methods of analysis based on vertical displacement as domain variable deal with solution of associated torsion problem in bending of plates. It is essential to use vertical displacement as face variable instead of domain variable in proper analysis of bending problems. It is useful to consider adapting and applying the present extended Poisson's theory for analysis of problems reported in [10, 11].

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