Exact Analysis of Raft Foundation Subjected to an Out-of-Plane Point Load

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Abstract

Foundation rafts, in various aspects behave as plates or slabs. This study focuses on the case of surface point load for both deflection and stress convergence of series solutions.Resting on an elastic foundation, the behavior under load is influenced by the modulus of the sub-grade reaction of the soil. Several methods of analysis have been developed in the past to obtain plate solution;however, in this study, the main focus is to use a new method of analysis, "the curvature-displacement method"to achieve point load solution. To achieve this, the shape function for an all-round simply supported plate was adopted. This is because the slab is assumed to rest simply on heavy perimeter beams. The curvature-displacement method was used to obtain the equation for deflection, curvature and moment using nine series-terms where convergence error became negligible. The equations obtained were thereafter, used to estimate the result for deflection, curvature and moment for a simplified solution. These are the results needed for design of the slab. Results found were among the best in the literature and the particular case of moments at the point load giving a convergent result of 0.43P stands out.

Keywords: *Raft foundation, Elastic foundation, Curvature and displacement, Modulus and sub-grade reaction*

I. INTRODUCTION

The use of plates has been important in engineering projects. They are widely used in different areas ranging from offshore platforms, ship deck, aircraft, building, chopper blades, retaining wall, floor slabs, concrete pavement etc. As a result of their wide use in different engineering fields such as civil, mechanical, aeronautical, naval among others, researchers continue to look into the subject for simpler yet near-exact solution.

Plates are defined as plane structural elements with a small thickness compared to the planar dimensions, (Timoshenko and Woinowsky, 1959).Many methods have been formulated in the past for the analysis of plates; some of them are the yield-line analysis by Johansen in 1972, the numerical methods of finite difference and finite element methods.The finite element methods involve dividing the physical systems, such as structures, solid or fluid continua, into small sub- regions or elements, (Roger, 1996;Zienkiewicz and Taylor, 1999) referred to this as discretization where each element is an essentially simple unit, the behaviour of which can be readily analyzed. Accurate modeling involves understanding the important relationships between the physical structure and the analytical simulation (James, 2005; Clough 1980)

The curvature-displacement method used in this research was developed by Prof. T. N. Johnarry (Johnarry, 2011). He noted that acceleration and displacement are the two variables that describe whether a structure is in simple harmonic state which indicates that there are no losses. The acceleration-displacement method was used by Johnarry, (2011) to solve plate problems.

A. Raft Foundation as a Plate

A plate is a structural element which is thin and flat. This means the thickness of the plate is small compared to the length and width dimensions.

$$\frac{t}{L} \ll 1$$

where t = thickness of plate

L= Length of plate

A raft foundation can be classified as a plate because it satisfies the expression above.

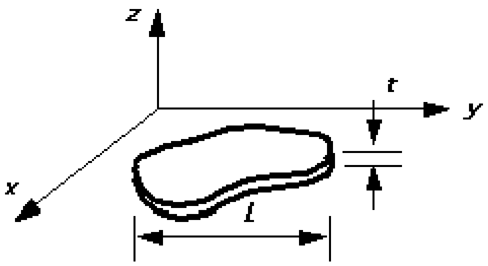

Figure 1: Plate and Associated (x, y,z) Coordinate System

When a plate (raft) is placed on an elastic foundation, there is usually a reaction from the sub grade. The problem can be simplified by assuming the intensity of the reaction of the sub grade is proportional to the deflection (w) of the plate. The intensity is given by the expression (kw) (Winkler, 1867). According to Timoshenko (1959), the value of k depends largely on the properties of the sub grade. He also provided a table for the values of the modulus of the subgrade ranging from very poor sub grade to best of base condition.

The differential equation governing plate on an elastic foundation is shown below

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q-kw}{D} \qquad 1$$

k = modulus of the foundation

The equation 1 above can be solved using Galerkin, Levy andNavier'ssolutions, depending on the boundary conditions. However, Johnarry (2011) solved the equation using the acceleration-deflection method. The method produced an exact solution for thin plate problem in transverse bending and obtained values for displacements, bending moments and buckling.

II. THEORETICAL FORMULATION

The basic theoretical formulations are described in the sub-headings that follow:

A. *Calculation of Deflection*

$$\text{Let } w = \sum\sum A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad 2$$

$$w_{xxxx} + w_{xxyy} + w_{yyyy} = \frac{q-kw}{D} \qquad 3$$

$$w_{xxxx} + w_{xxyy} + w_{yyyy} + \frac{kw}{D} = \frac{q}{D} \qquad 4$$

$$Let\ S = \frac{k}{D} \qquad 5$$

Solving the LHS of equation 2, we have;

$$A_{mn}\left[\frac{\pi^6}{16}\right] mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 = H \qquad 6$$

Therefore, equation 2.2 becomes

$$H + Sw = \frac{q}{D} \qquad 7$$

However, we can simplify Sw as follow:

$$Sw = \frac{\int_0^b \int_0^a Sw.w\partial x\partial y}{\int_0^b \int_0^a w\partial x\partial y} \qquad 8$$

$$Sw = \frac{SA_{mn}\, mn\pi^2}{16} \qquad 9$$

Equation 2.5 becomes

$$A_{mn}\left[\frac{\pi^6}{16}\right] mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 + \frac{SA_{mn}\, mn\pi^2}{16} = \frac{q}{D} \qquad 10$$

In the case of a point load, q^* only exists over an isolated centre area where the total load is P.

$$q^* = \frac{q^*}{1} = \frac{\int_0^b \int_0^a q^*.w\partial x\partial y}{\int_0^b \int_0^a w\partial x\partial y} = \frac{q^*\partial x\partial y \int_0^b \int_0^a .w}{\int_0^b \int_0^a w\partial x\partial y} \qquad 11$$

$q^*\ absorbs\ \partial x\partial y\ and\ becomes\ P$. The equation therefore results into

$$= \frac{P \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{4ab/mn\pi^2} \qquad 12$$

Combining the LHS and the RHS of equation 2.5 and solving, we have

$$A_{mn} = \frac{P \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}}{\left[\frac{1}{16}\right]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + S\right]4Dab} \qquad 13$$

$$w = A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad 14$$

Inserting A_{mn} in w, we have

$$w = \frac{4P\left(\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + S\right]} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad 15$$

The above equation is an exact equation similar to that obtained by Timoshenko (1959), hence

$$w = \frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}}{\left[\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + k\right]} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad 16$$

$\left(\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\right)_p$ is the location function at the point where the point load is acting.

If the point load is placed at the middle of the plate, i.e. $x = 0.5a, y = 0.5b$ as shown in figure 2 and 3 below.

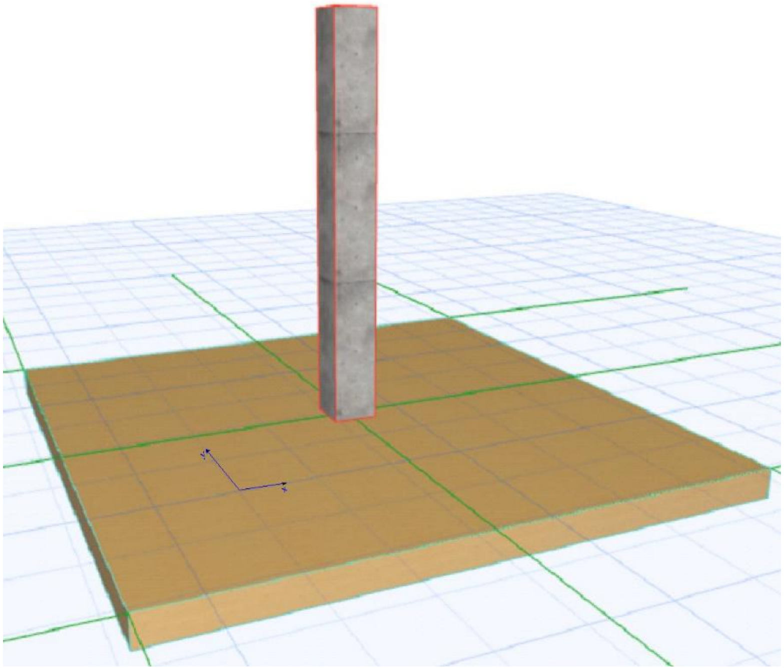

Figure 2: Plate with a Surface Point Load

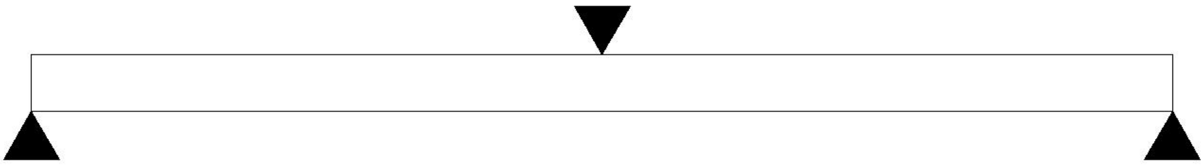

Figure 3: A Section Showing the Structural Diagram

For simplicity, we take $a = b$ which is the case of a square plate

Let $\gamma = \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2$

We can get the values of γ at 1,1, 1,3, 3,1, …., 5,5 for m,n

B. *Calculation of Curvature*

From the deflection equation 2.13that was derived above we have,

$$\mathrm{w} = \frac{4P\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)_p$ represents the location function for the load. To differentiate to calculate curvature, we have two different scenarios

1. When differentiating the deflection to obtain the curvature at the position where the point load is acting, the location function $\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)_p$, participates in the differentiation.
2. On the other hand, when differentiating the deflection to obtain the curvature at points outside the position of the point load, $\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)_p$, does not participate in the differentiation.

The following steps show how the derivation of the curvature for the plate along different points on a typical axis is obtained.

1) *Curvature Along X-axis*

For better understanding, let us represent the location function as follow

$x = \xi, y = \eta$ at the point where the load is acting.

Therefore, the deflection equation becomes

$$\mathrm{w} = \frac{4P\left(\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad 17$$

$$\mathrm{w}_x^{u} = \chi_x = \frac{-4P\left(\frac{m^2\pi^2}{a^2}\right)\left(\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad 18$$

Similarly,

$$w_y^{ll} = \chi_y = \frac{-4P\left(\frac{n^2\pi^2}{b}\right)\left(\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad 19$$

The curvature at the point where the point load is acting is calculated as follow

$$w = \frac{4P\sin^2\frac{m\pi x}{a}\sin^2\frac{n\pi y}{a}}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]} \quad 20$$

$$w_x^{ll} = \chi_x = \frac{8P\left(\frac{m^2\pi^2}{a^2}\right)\left(\cos\frac{2m\pi x}{a}\sin^2\frac{n\pi y}{b}\right)}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]} \quad 21$$

Similarly, $w_y^{ll} = \chi_y = \frac{8P\left(\frac{n^2\pi^2}{b^2}\right)\left(\sin^2\frac{m\pi x}{a}\cos\frac{2m\pi y}{b}\right)}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]}$ 22

C. Calculation of Moments

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = -D\left[\frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2}\right]$$

$$M_{xy} = -M_{-yx} = D(1-v)\frac{\partial^2 w}{\partial x \partial y}$$

Therefore, we can say

$$M_x = -D[\chi_x + v\chi_y]$$
$$M_y = -D[\chi_y + v\chi_x]$$

For a square plate, $\chi_x = \chi_y$, we can simplify moment as follow

$$M_x = -D[\chi_x + v\chi_y]$$
$$M_x = -D\chi_x[1+v]$$
$$M_y = -D\chi_y[1+v]$$

If we consider a simple plate of dimension $4m \times 4$m with a thickness of $200mm$ resting on an elastic foundation. A column supporting a load of $1500kN$ at ultimate limit state is acting at the centre of the plate.

$E = 21 \times 10^9 \text{kN/m}^2$, $D = \frac{21\times10^9\times0.2^3}{12(1-0.2^2)} = 15385kNm$

Using Biot's equation for calculation of k, we have

$$k = 0.65\frac{E_s}{1-v_s^2}\left[\frac{E_s B^4}{E_b I}\right]^{1/12} = 5022kN/m^2/m$$

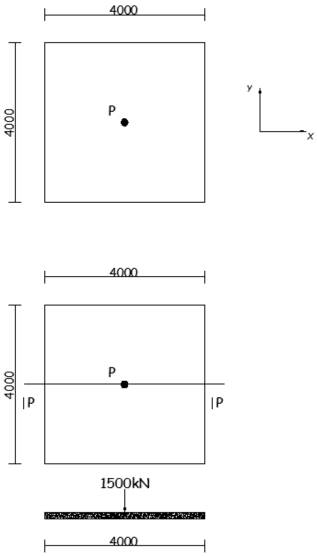

Figure 4: Plan and Section of a Simplified Plate

III. RESULTS

The results obtained from the calculation for deflection, curvature and moment are presented in a number of tables and graphs in this section.

A. Deflection Values

Table 1: Deflection at Mid Span

x	$\delta(m) \times 10^{-3}$	$w\ (m) \times 10^{-3}$
0	0	0.000
0.1a	4.628	3.758
0.2a	8.988	7.336
0.3a	13.100	10.800
0.4a	16.480	13.740
0.5a	17.810	14.980
0.6a	16.480	13.740
0.7a	13.100	10.800
0.8a	8.988	7.336
0.9a	4.628	3.758
a	0.00	0.000

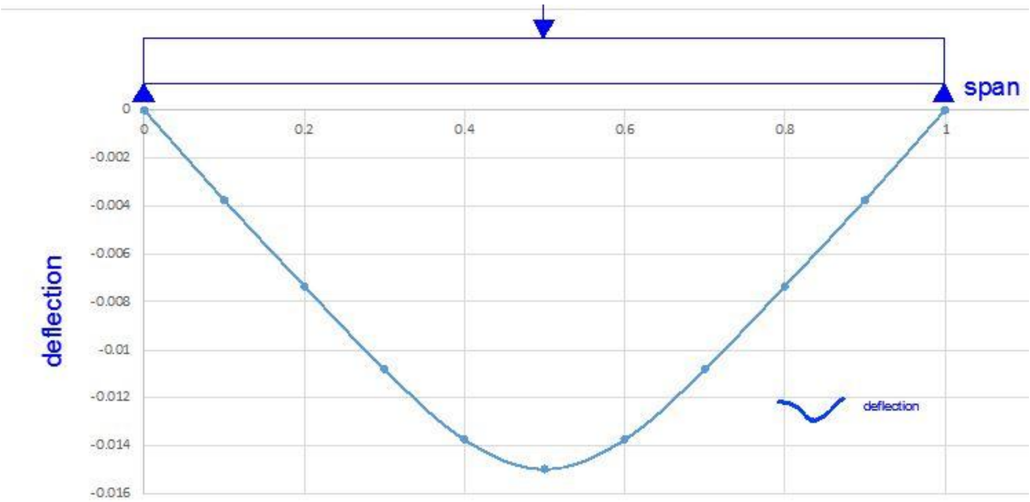

Figure 5: Deflection of Plate on Elastic Foundation with the Influence of Foundation Modulus

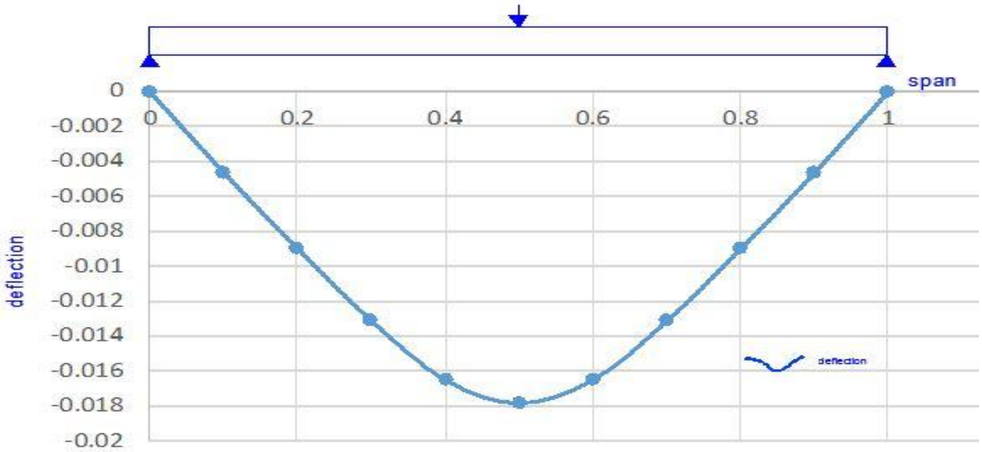

Figure 6: Deflection of Plate Without the Influence of Foundation Modulus

B. Curvature Values

Table 2: Curvature at Mid Span without Foundation Modulus

x	$\boldsymbol{\sigma}$
0	0
$0.1a$	$\frac{-0.01938P}{D}$
$0.2a$	$\frac{-0.0246P}{D}$
$0.3a$	$\frac{-0.0422P}{D}$
$0.4a$	$\frac{-0.131P}{D}$
$0.5a$	$\frac{-0.3693P}{D}$
$0.6a$	$\frac{-0.131P}{D}$
$0.7a$	$\frac{-0.0422P}{D}$

$0.8a$	$\frac{-0.0246P}{D}$
$0.9a$	$\frac{-0.01938P}{D}$
a	0

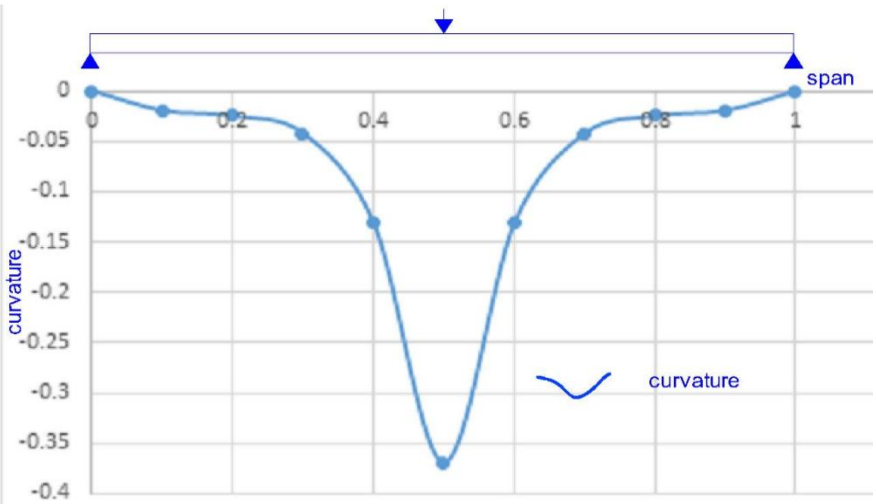

Figure 7: Plot of Curvature of Plate without the Influence of Foundation Modulus

Table 3: Curvature at Mid Span with the Effect of Foundation Modulus

x	$\chi_x \times 10^{-3}$
0	0
0.1a	−1.374
0.2a	−0.359
0.3a	−2.714
0.4a	−11.100
0.5a	32.440
0.6a	−11.100
0.7a	−2.714
0.8a	−0.359
0.9a	−1.374
a	0

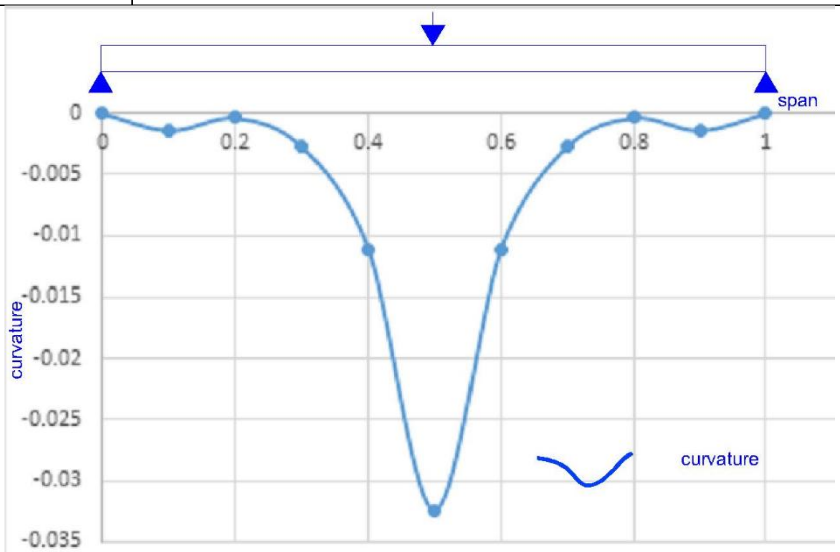

Figure 8: Plot of Curvature of Plate Under the Influence of Foundation Modulus

C. *Moment Values*

Table 4: Moment at Mid-Span

x	M
0	0.000
0.1a	27.480
0.2a	7.186
0.3a	54.280
0.4a	222.010
0.5a	648.820
0.6a	222.010
0.7a	54.280
0.8a	7.186
0.9a	27.480
a	0.000

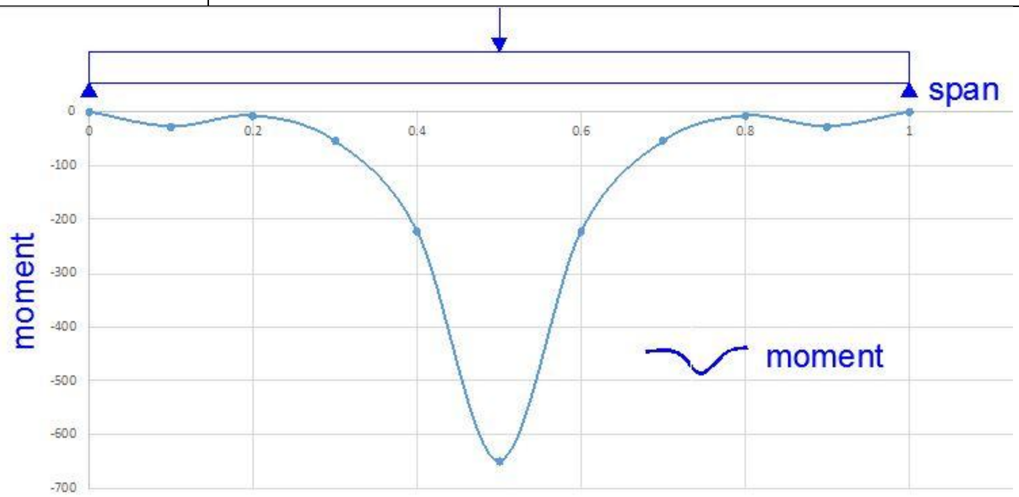

Figure 8: Plot of Moment at Mid-Span

D. *Discussion of Results*

From the calculation in the preceding sections, the deflection component was separated into two, namelyδ, which is the deflection without the influence of foundation modulus kand w, the deflection with the influence of foundation modulus. From the result obtained from deflection values, we can see that the value for the maximum deflection (δ), $0.01142\frac{Pa^2}{D}$ is close to the value $\frac{0.0116Pa^2}{D}$obtained by Timoshenko and Woinowsky-Krieger (1959), for plate with a point load. With the inclusion of modulus of subgrade reaction which is the main focus of the paper, it was observed that the equation derived using the curvature-displacement method $\left[w = \frac{4P\left(\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]$is similar to that obtained byTimoshenko and Woinowsky-Krieger (1959)in the form$\left[\frac{4P}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}}{\left[\pi^4 D\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+k\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]$.

The solution of the deflection equation gives $\frac{0.009603Pa^2}{D}$. Also, it was observed that the maximum deflection occurs at the centre of the plate for both cases (with and without the influence of foundation modulus). It was again observed from the calculation of deflections that $\delta > w$, i.e. the effect of foundation modulus reduces the deflection of plate under point load. Differentiating the deflection equation twice, the curvature equation was obtained at the point within the vicinity of the point load $\chi_x = \left[\frac{8P\left(\frac{m^2\pi^2}{a^2}\right)\left(\cos\frac{2m\pi x}{a}sin^2\frac{n\pi y}{b}\right)}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]}\right]$ and at points away from the point load as$\left[\chi_x = \frac{-4P\left(\frac{m^2\pi^2}{a^2}\right)\left(\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\right)_p}{[Dab]\left[\pi^4\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2+S\right]}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\right]$.

In engineering design, the value of bending moment is very important as this allows us to provide adequate reinforcement for the part of the structure that is being designed. Several attempts have been made to evaluate the maximum moment under a point load with some authors making assumptions to obtain a value. Timoshenko and Woinowsky (1959) made an attempt at obtaining the solution, but their series did not converge. They made an assumption that the point load acts on the plate through a column of dimension $0.1a$, which is10% of the plate dimension and obtained a value of $0.298P$.

In this research, a total of 9 series was considered until convergence error became negligible. With this a value of $0.43P$ was obtained for the moment. To achieve this result, a location function was assumed, which takes care of points within the region of the point load.

IV. CONCLUSION

The derivation of the equation for deflection, curvature and moment was carried out using the curvature-displacement method to solve the plate problem.

The modulus of subgrade reaction, *k*, can be obtained from plate load bearing test. However, for the purpose of this thesis Biot's equation was adopted to obtain the *k* values.

From the results obtained the following specific conclusion can be made:

1. Curvature-displacement method can provide an exact solution for plate with a point load on an elastic foundation.
2. Faster convergence rate was obtained with the 9 series that was used.
3. A deflection value of $\frac{0.009603Pa^2}{D}$ was obtained while for the moment, a value of 0.43P was obtained using the method.

A. Recommendation

The type of loading considered in this research was a static point load as dynamic loading was not considered. The application of the curvature-displacement method of analysis to dynamic load cases is a potential topic for further research.

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