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Simple and Exact Approach to PostBuckling Analysis of Rectangular Plate

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Abstract
This paper presents a new, simple, and exact approach to the post-buckling analysis of thin rectangular plates. In the stud, the Airy's stress functions are not incorporated as the middle surface axial displacement equations are determined as direct functions of middle surface deflection. With this, the bending and membrane stresses and strains, which are direct functions of middle surface deflection are obtained. These stresses and strains are used to obtain the total potential energy functional. The minimization of the total potential energy gives the governing equation and compatibility equation for rectangular thin plates buckling with large deflection. The compatibility equations and the governing equation are solved to obtain the deflection function for the problem. Direct variation is applied to the total potential energy function to get the formula to calculate the buckling loads. A numeric analysis is performed for a plate with all the four edges simply-supported (SSS plate). It is observed that when the deflection to thickness ratio (w/t) is zero the buckling load obtained coincides with the critical buckling from small deflection (linear) analysis. Another observation is that the values of buckling load for given values of w/t obtained in the present study do not vary significantly with those obtained by Samuel Levy. The recorded average percentage difference is 12.65%. It is also observed that the maximum w/t to be considered when small deflection analysis is to be used is 0.225. When w/t is more than 0.225, using small deflection analysis will give erroneous results. Thus, a large deflection analysis is recommended when w/t is above 0.25. We conclude and recommend that this new equation for the analysis of thin plates is a better alternative to the popular von Karman equation.

Keywords: *Post-buckling buckling load, membrane strains, total potential energy, minimization, direct variation*

List of notations

$\varepsilon_{xx}, \varepsilon_{yy} =$ Nonlinear strains along x and y direction respectively, γ_{xy} = Inplane shear strain

u_0 *and* $u_0 =$ Middle surface displacement along x and y direction respectively

u and v = Nonlinear in-plane displacement,D = flexural rigidity, $\boldsymbol{\nu}$ = Poisson ratio

Π= Total potential energy functional, $k_{ij} =$ Plate stiffness,

$N_x$andσ_x are Buckling/Postbuckling load and stress respectively

w is displacement in the z-direction, A is the amplitude of deflection, Ƨ is the aspect ratio (b/a)

I. INTRODUCTION

Large deflection analysis of rectangular plate hinges mostly of von-Karman type nonlinear strain-displacement relation [1], which is given herein as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x}\right] \quad (1)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y}\right] \quad (2)$$

The complex nature of the terms in the square bracket, which sum to total membrane strain of the plate along x or y direction, as the case may be, is the major constraint encountered in large deflection analysis of plate. From most of the works studied, little effort had been made to determine the expression for u_0 and v_0. Most of them end up assuming (instead of determining) a function for u_0 and v_0 ([2], [3],[4],[5], [6],[7][8]). Another, complex issue in the analysis of a plate with large deflection is the issue of Airy's stress function.

Earlier scholars had assumeAiry's stress function without determining them through the integration of the governing equation and compatibility equation. However, some recent scholars [9],[10][11], in the various Ph.D. theses determined the stress functions they used in their works (post-buckling, forced vibration, and pure bending analyses of rectangular plates with large deflection). A critical study of their works revealed how involving and lengthy the expression for stress functions is. The nature of the stress function determined by them can discourage any engineering analyst.

In order, to circumvent the use of Airy's stress function and avoid arriving at the same governing equation introduced by von-Karman, this study presents a simple and exact approach to the analysis of rectangular thin plates with large deflection.

II. THEORETICAL ANALYSIS

A. Middle Surface Displacements

The major assumption of the analysis of plates with large deflection is that the middle surface displacements are not zeros. The first step in determining the middle surface displacements is to determine their nature. To do this, consider membrane terms of equations 1 and 2:

$$\varepsilon_{xxm} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x} \quad (3)$$

$$\varepsilon_{yym} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y} \quad (4)$$

Minimizing equations 3 and 4 gives:

$$u_0 = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad (5)$$

$$v_0 = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \quad (6)$$

The coefficient of equations 5 and 6 is minus a half. This is the coefficient that minimizes equations 3 and4. Thus, any other coefficient (which is not minus) of equations 5 and 6 shall make equations 3 and 4 not to become zeros. However, there is a need to determine the optimum value of the coefficient whenever the plate loses its bending stiffness and carries the load with the help of only membrane resistance. He [12], was able to determine this coefficient. He did this by replacing the minus half with an arbitrary constant to get:

$$u_0 = c_1\left(\frac{\partial w}{\partial x}\right)^2 \quad (7)$$

$$v_0 = c_1\left(\frac{\partial w}{\partial y}\right)^2 \quad (8)$$

Substituting equations 7 and 8 into equations 3 and 4 gave:

$$\varepsilon_{xxm} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + c_1\left(\frac{\partial w}{\partial x}\right)^2 = c_2\left(\frac{\partial w}{\partial x}\right)^2 \quad (9)$$

$$\varepsilon_{yym} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + c_1\left(\frac{\partial w}{\partial y}\right)^2 = c_2\left(\frac{\partial w}{\partial y}\right)^2 \quad (10)$$

$$Where: \quad c_2 = c_1 + \frac{1}{2} \quad (11)$$

Substituting equation 9 into equation 1 gives:

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} + c_2\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\partial^2}{\partial x^2}(-zw + c_2 w^2) \quad (12)$$

Extremizinng equation 12 with respect w gives:

$$\frac{\partial \varepsilon_{xx}}{\partial w} = \frac{\partial^2}{\partial x^2}(-z + 2c_2 w) = 0. \quad That\ is:$$

$$c_2 = \frac{z}{2w} \quad (13)$$

The extreme strain occurs at the outermost fiber where z = t/2. Thus:

$$c_2 = \frac{\frac{t}{2}}{2w} = \frac{1}{4}.\frac{t}{w} \quad (14)$$

Assuming here that the extreme strain occurs when t/w is up to unity and above. Thus:

$$c_2 = \frac{1}{4} \quad (15)$$

Substituting equation 15 into equation 11 gives:

$$\frac{1}{4} = c_1 + \frac{1}{2}. \ That\ is:$$

$$c_1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \quad (16)$$

Substituting equation 16 into equations 7 and 8 gives:

$$u_0 = -\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2 \quad (17)$$

$$v_0 = -\frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2 \quad (18)$$

B. Nonlinear in-plane displacements

Integrating equations 1 and 2 concerning x and y respectively gives:

$$u = -z\frac{\partial w}{\partial x} + \left[\frac{1}{2}\frac{\partial w^2}{\partial x} + u_0\right] \quad (20)$$

$$v = -z\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{2}\frac{\partial w^2}{\partial y} + v_0\right] \quad (21)$$

Substituting equations 17 and 18 into equations 20 and 21 respectively gives:

$$u = -z\frac{\partial w}{\partial x} + \left[\frac{1}{2}\frac{\partial w^2}{\partial x} - \frac{1}{4}\frac{\partial w^2}{\partial x}\right] = -z\frac{\partial w}{\partial x} + \frac{1}{4}\frac{\partial w^2}{\partial x} \quad (22)$$

$$v = -z\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{2}\frac{\partial w^2}{\partial y} - \frac{1}{4}\frac{\partial w^2}{\partial y}\right] = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{4}\frac{\partial w^2}{\partial y} \quad (23)$$

C. Nonlinear Strain displacement relations

Differentiating equations 22 and 23 concerning x and y gives respectively:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^2 \quad (24)$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2 \quad (25)$$

The in-plane shear strain within x - y plane is:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\left[-z\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \quad (26)$$

D. Total potential energy Functional

The total potential energy of a thin rectangular plate under buckling is given as:

$$\Pi = \frac{1}{2}\iiint\left(\sigma_{xx}.\varepsilon_{xx} + \sigma_{yy}.\varepsilon_{yy} + \tau_{xy}.\gamma_{xy}\right) dx\,.dy\,.dz - \frac{N_x}{2}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dx\,dy \quad (27)$$

Substituting the constitutive relation into equation 27 gives:

$$\Pi = \frac{E}{2(1-\mu^2)}\iiint\left(\varepsilon_{xx}^2 + 2\mu\varepsilon_{xx}.\varepsilon_{yy} + \varepsilon_{yy}^2 + (1-\mu)\frac{\gamma_{xy}^2}{2}\right) dx\,.dy\,.dz - \frac{N_x}{2}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dx\,dy \quad (28)$$

Substituting equations 24, 25, and 26 into equation 28 give equation 29.

$$\Pi = \frac{E}{2(1-\nu^2)}\int_0^a\int_0^b\int_{-t/2}^{t/2}\left\{z^2\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] - \frac{z}{2}\left[\frac{\partial^2 w}{\partial x^2}.\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial^2 w}{\partial x \partial y}.\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) + \frac{\partial^2 w}{\partial y^2}.\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1}{16}\left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right]\right\} dx\,.dy\,.dz - \frac{N_x}{2}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dx\,dy \quad (29)$$

Carrying out the closed domain integration of equation 29 concerning z gives:

$$\Pi = \frac{D}{2}\iint\{\left[\left(\frac{d^2 w}{dx^2}\right)^2 + 2\left(\frac{d^2 w}{dxdy}\right)^2 + \left(\frac{d^2 w}{dy^2}\right)^2\right] dx\,.dy + \frac{\mathrm{gD}}{2\times 16}\iint\left(\left[\frac{\partial w}{\partial x}\right]^4 + 2\left[\frac{\partial w}{\partial x}\right]^2\left[\frac{\partial w}{\partial y}\right]^2 + \left[\frac{\partial w}{\partial y}\right]^4\right) dx\,.dy - \frac{N_x}{2}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dx\,dy \quad (30a)$$

$$Where: D = \frac{\mathrm{Et}^3}{12(1-\nu^2)};\; g = \frac{12}{\mathrm{t}^2};\; gD = \frac{12}{\mathrm{t}^2}\times\frac{\mathrm{Et}^3}{12(1-\nu^2)} = \frac{\mathrm{Et}}{(1-\nu^2)} \quad (30b)$$

Equation 30a can be written in terms of the non-dimensional coordinates (R = x/a and Q = y/b) as:

$$\Pi = \frac{bD}{2a^3}\iint\left[\left(\frac{d^2 w}{dR^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{d^2 w}{dRdQ}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{d^2 w}{dQ^2}\right)^2\right] dR\,dQ$$
$$\frac{\mathrm{bgD}}{2a^3\times 16}\iint\left[\left(\frac{\partial w}{\partial R}\right)^4 + \frac{2}{Ƨ^2}\left(\frac{\partial w}{\partial R}\right)^2\left(\frac{\partial w}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial w}{\partial Q}\right)^4\right] dR\,dQ - \frac{bN_x}{2a}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dR\,dQ \quad (31)$$

Minimizing equation 30a with respect to w, u_0 and v_0 give the governing equation and two displacement compatibility equations as presented on equations 32, 33, and 34 respectively. In this case, minimization concerning u_0 and v_0 shall be based on the differential part (excluding the coefficient c_1). In this regard, equation 30 shall be rewritten as equation 30c:

$$\Pi = \frac{D}{2}\iint\{\left[\frac{d^3}{dx^3}\frac{dw^2}{dx} + \frac{d^3}{dxdy^2}\frac{dw^2}{dx} + \frac{d^3}{dx^2dy}\frac{dw^2}{dy} + \frac{d^3}{dy^3}\frac{dw^2}{dy}\right]dx\,.dy\ \frac{\mathrm{gD}}{2\times 16}\iint\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial w^2}{\partial x}\right)^2 + \frac{\partial}{\partial x}.\frac{\partial}{\partial x}.\frac{\partial}{\partial y}.\frac{\partial}{\partial y}(w^2)^2 + \frac{\partial^2}{\partial y^2}\left(\frac{\partial w^2}{\partial y}\right)^2\right)dx\,.dy - \frac{N_x}{2}\iint\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 dx\,dy \qquad (30c)$$

Minimizing equation 30c concerning w gives:

$$\frac{\partial\Pi}{\partial w} = D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \frac{2gD}{16}\left(\left[\frac{\partial w}{\partial x}\right]^2\frac{\partial^2 w}{\partial x^2} + \left[\frac{\partial w}{\partial y}\right]^2\frac{\partial^2 w}{\partial x^2} + \left[\frac{\partial w}{\partial x}\right]^2\frac{\partial^2 w}{\partial y^2} + \left[\frac{\partial w}{\partial y}\right]^2\frac{\partial^2 w}{\partial y^2}\right) + N_x\frac{\partial^2 w}{\partial x^2} = 0 \qquad (32)$$

Minimizing equation 30c concerning (dw^2/dx) gives:

$$\frac{\partial\Pi}{\partial\left(\frac{\partial w^2}{\partial x}\right)} = \frac{D}{2}\frac{\partial}{\partial x}\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right] + \mathrm{gD}c_2^2\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - 0 = 0$$

That is:

$$\frac{\partial\Pi}{\partial\left(\frac{\partial w^2}{\partial x}\right)} = \frac{\mathrm{gD}}{16}.\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

That is:

$$\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0 \qquad (33)$$

Minimizing equation 30c concerning (dw^2/dy) gives:

$$\frac{\partial\Pi}{\partial\left(\frac{\partial w^2}{\partial y}\right)} = \frac{D}{2}\frac{\partial}{\partial y}\left[\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right] + \frac{\mathrm{gD}}{16}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] - 0 = 0$$

That is:

$$\frac{\partial\Pi}{\partial\left(\frac{\partial w^2}{\partial y}\right)} = \frac{\mathrm{gD}}{16}.\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

That is:

$$\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0 \qquad (34)$$

From equations 33 and 34, it is gathered that:

$$\left(\frac{\partial w}{\partial x}\right)^2 = -\left(\frac{\partial w}{\partial y}\right)^2 \qquad (35)$$

Strains of middle surface ($\varepsilon_{x0} and\ \varepsilon_{x0}$) of the plate are:

$$\varepsilon_{x0} = \frac{\partial u_0}{\partial x} = -\frac{1}{3}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (36)$$

$$\varepsilon_{y0} = \frac{\partial u_0}{\partial x} = -\frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (37)$$

Substituting equation 35 into equation 36 gives:

$$\varepsilon_{x0} = \frac{\partial u_0}{\partial x} = \frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (38)$$

Comparing equations 37 and 38 reveals that:

$$\varepsilon_{x0} = -\varepsilon_{y0} \qquad (39)$$

Substituting equation 35 into equation 32gives:

That is:

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + N_x\frac{\partial^2 w}{\partial x^2} = 0 \qquad (40)$$

The solution of equation 40 is in the trigonometric form is given as:

$$w = [a_0 \quad a_1 \quad a_2 \quad a_3]\begin{bmatrix}1\\ R\\ \cos kR\\ \sin kR\end{bmatrix} \times [b_0 \quad b_1 \quad b_2 \quad b_3]\begin{bmatrix}1\\ Q\\ \cos gQ\\ \sin gQ\end{bmatrix} \qquad (41)$$

From equations 41 is gathered that:

$$w = a_i h_x \times b_i h_y = Ah \qquad (42)$$

Substituting equation 42 into equation 31gives equation 43:

$$\Pi = \frac{A^2 bD}{2a^3}\iint\left[\left(\frac{d^2h}{dR^2}\right)^2 + \frac{2}{\text{Ƨ}^2}\left(\frac{d^2h}{dRdQ}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{d^2h}{dQ^2}\right)^2\right]dR\,dQ + \frac{A^4\mathrm{bgD}}{32a^3}\iint\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial h}{\partial Q}\right)^4\right]dR\,dQ - \frac{A^2 bN_x}{2a}\iint\left(\frac{\mathrm{dh}}{\mathrm{dR}}\right)^2 dR\,dQ \qquad (43)$$

Minimizing equation 43 concerning A gives:

$$\frac{\partial \Pi}{\partial \mathrm{A}} = \frac{AbD}{a^3}\iint\left[\left(\frac{d^2h}{dR^2}\right)^2 + \frac{2}{\text{Ƨ}^2}\left(\frac{d^2h}{dRdQ}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{d^2h}{dQ^2}\right)^2\right] dR\, dQ$$

$$\frac{A^3\mathrm{bgD}}{8a^3}\iint\left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial h}{\partial Q}\right)^4\right] dR\, dQ \; - \frac{AbN_x}{2a}\iint\left(\frac{\mathrm{dh}}{\mathrm{dR}}\right)^2 dR\, dQ = 0 \qquad (44)$$

Equation 44 shall be written in a symbolized form as:

$$\left(k_{bx} + \frac{2k_{bxy}}{\text{Ƨ}^2} + \frac{k_{by}}{\text{Ƨ}^4}\right) + \frac{A^2\mathrm{g}}{8}\left(k_{mx} + \frac{2k_{mxy}}{\text{Ƨ}^2} + \frac{k_{my}}{\text{Ƨ}^4}\right) - \frac{N_x a^2}{D}k_{Nx} = 0 \qquad (45)$$

That is:

$$k_{bT} + \frac{A^2\mathrm{g}}{8}(k_{mT}) - \frac{N_x a^2}{D}k_{Nx} = 0 \qquad (46)$$

Where:

$$k_{bx} = \iint\left(\frac{d^2h}{dR^2}\right)^2 dR\, dQ;$$

$$k_{bxy} = \iint\left(\frac{d^2h}{dR\, dQ}\right)^2 dR\, dQ;$$

$$k_{by} = \iint\left(\frac{d^2h}{dQ^2}\right)^2 dR\, dQ$$

$$k_{mx} = \iint\left(\frac{\partial h}{\partial R}\right)^4 dR\, dQ;$$

$$k_{mxy} = \iint\left(\frac{\partial h}{\partial R}\right)^2\left(\frac{\partial h}{\partial Q}\right)^2 dR\, dQ;$$

$$k_{my} = \iint\left(\frac{\partial h}{\partial Q}\right)^4 dR\, dQ$$

$$k_{Nx} = \iint\left(\frac{dh}{dR}\right)^2 dR\, dQ;$$

$$k_{bT} = k_{bx} + \frac{2k_{bxy}}{\beta^2} + \frac{k_{by}}{\beta^4};$$

$$k_{mT} = k_{mx} + \frac{2k_{mxy}}{\beta^2} + \frac{k_{my}}{\beta^4}$$

Substituting equations for g and D into equation 30b gives:

$$k_{bT} + 2A^2 \times \frac{12}{\mathrm{t}^2} \times \frac{1}{16}(k_{mT}) - \frac{12(1-\nu^2)}{\mathrm{Et}^3}N_x a^2 k_{Nx} = 0\ .\ \mathit{That\ is}:$$

$$k_{bT} + \frac{3A^2}{2\mathrm{t}^2}.(k_{mT}) - 12(1-\nu^2)k_{Nx}\frac{N_x a^2}{\mathrm{Et}^3} = 0.\ \mathit{After\ rearranging\ gives}:$$

$$k_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2(k_{mT}) = 12(1-\nu^2)k_{Nx}\frac{N_x a^2}{\mathrm{Et}^3} \qquad (47a)$$

$$k_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2(k_{mT}) = \frac{N_x a^2}{\mathrm{D}}k_{Nx} \qquad (47b)$$

$$k_{bT} + \frac{3}{2}\left(\frac{A}{t}\right)^2(k_{mT}) = 12(1-\nu^2)k_{Nx}\frac{\sigma_x a^2}{\mathrm{Et}^2} \qquad (47c)$$

$$\mathit{Where}: \sigma_x = \frac{N_x}{\mathrm{t}}$$

Rearranging equation 47c and making deflection coefficient to thickness ratio the subject of the formula gives:

$$\left(\frac{A}{t}\right)^2 = 8(1-\nu^2) * \frac{k_{Nx}}{k_{mT}}\frac{\sigma_x a^2}{\mathrm{Et}^2} - \frac{2}{3}\frac{k_{bT}}{k_{mT}} \qquad (48)$$

Rearranging equation 47a and making buckling load parameter the subject of the formula gives:

$$\frac{N_x a^2}{\mathrm{Et}^3} = \frac{1}{12(1-\nu^2)}\left[\frac{k_{bT}}{k_{Nx}} + \frac{3}{2}\left(\frac{A}{t}\right)^2 * \frac{k_{mT}}{k_{Nx}}\right] \qquad (49)$$

Rearranging equation 47c and making stress parameter the subject of the formula gives:

$$\frac{\sigma_x a^2}{\mathrm{Et}^2} = \frac{1}{12(1-\nu^2)}\left[\frac{k_{bT}}{k_{Nx}} + \frac{3}{2}\left(\frac{A}{t}\right)^2 * \frac{k_{mT}}{k_{Nx}}\right] \qquad (50)$$

Equations (49) and (50) are the load and stress parameters respectively, from which the post-buckling (N_x) and post-stress (σ_x) equations can be obtained.

E. Numerical Example

Analyze the post-buckling of a thin rectangular plate with all the four edges simply supported. The Poisson's ratio (ν)is 0.316.

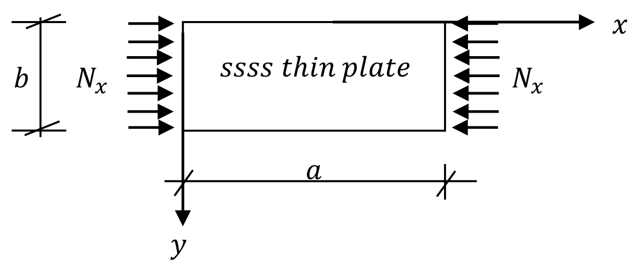

Figure 1: Sketch of SSSS plate with in-plane load along the x-direction

SSSS plate with shape function,

$$h = (Sin\ \pi R)(Sin\ \pi Q);$$

$$h_x = (Sin\ \pi R)\ ;\ \ h_y = (Sin\ \pi Q)$$

Substituting this into the stiffness equation and evaluate the integrals we have the stiffness as

$$k_{bx} = \frac{\pi^4}{4};\ \ k_{bxy} = \frac{\pi^4}{4};\ \ k_{by} = \frac{\pi^4}{4};$$

$$k_{mx} = \frac{9\pi^4}{64};\ \ k_{mxy} = \frac{\pi^4}{64};\ k_{my} = \frac{9\pi^4}{64};$$

$$k_{bT} = \frac{\pi^4}{4}\left(1 + \frac{2}{\text{Ƨ}^2} + \frac{1}{\text{Ƨ}^4}\right);$$

$$k_{mT} = \frac{\pi^4}{64}\left(9 + \frac{2}{\text{Ƨ}^2} + \frac{9}{\text{Ƨ}^4}\right);\quad k_{Nx} = \frac{\pi^2}{2}$$

When these stiffness are substituted into equation 49, the results obtained for the buckling/post-buckling coefficient, η, are presented in Table 1.

III. RESULTS AND DISCUSSIONS

The buckling and post-buckling loads for the thin SSSS rectangular plate of various aspect ratios obtained from equation 49 are presented in Table 1. It is easily noticed that when the deflection to thickness ratio is zero, the values obtained coincide with the critical buckling loads of the plate. This is quite acceptable because the critical buckling load occurs just before buckling is experienced. It is also observed that as the aspect ratio increase, the buckling and post-buckling loads decrease. Again, as the w/t ratio increases the post-buckling load increases.

The result obtained in this work for square SSSS plate is compared with the work of Samuel Levy [2]. This is presented in Table 2. It is observed that the highest percentage difference recorded is 22%. This occurred at the load parameter ($\boldsymbol{\sigma}_x$ a^2 / (Et^2)) of 3.72. Another glaring observation made in Table 2 is that as the load increase up to 21.45, the percentage difference decreases down to 4.53%. On average, the percentage difference is 12.65%. This percentage difference recorded here may be attributed to the different approaches adopted by the present authors and Levy. Even though the difference is somewhat high, the simplicity of the present approach may not be overlooked. Hence, one can confidently use the present approach to analyze thin rectangular plates with large deflection of various boundary conditions.

Table 1: Buckling and post-buckling loads coefficient (η) of SSSS thin plate

w/t	$\eta = \frac{N_x a^2}{D}$										
	Aspect ratio, Ƨ = b/a										
	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
0	39.48	32.92	28.34	25.01	22.51	20.59	19.09	17.88	16.90	16.09	15.42
0.25	40.64	33.90	29.19	25.78	23.22	21.27	19.73	18.50	17.51	16.69	16.00
0.498	44.07	36.78	31.72	28.07	25.35	23.27	21.65	20.35	19.31	18.45	17.73
0.743	49.69	41.51	35.86	31.82	28.82	26.55	24.78	23.38	22.25	21.33	20.56
0.984	57.40	47.98	41.53	36.95	33.59	31.04	29.08	27.53	26.29	25.27	24.44
1.22	67.02	56.06	48.62	43.37	39.54	36.66	34.45	32.71	31.33	30.20	29.28
1.45	78.39	65.61	56.99	50.95	46.56	43.29	40.79	38.83	37.28	36.02	35.00
1.673	91.27	76.43	66.48	59.54	54.53	50.81	47.97	45.77	44.03	42.63	41.48
1.889	105.51	88.39	76.97	69.03	63.33	59.11	55.91	53.44	51.49	49.92	48.64
2.101	121.17	101.54	88.50	79.47	73.01	68.24	64.65	61.87	59.68	57.94	56.52
2.303	137.63	115.37	100.62	90.44	83.18	77.85	73.83	70.73	68.31	66.37	64.80
2.498	154.95	129.92	113.38	102.00	93.89	87.95	83.49	80.06	77.38	75.24	73.52
2.687	173.09	145.16	126.73	114.09	105.10	98.53	93.60	89.83	86.88	84.53	82.64
2.871	192.01	161.05	140.67	126.70	116.80	109.57	104.16	100.02	96.79	94.23	92.17
3.044	210.95	176.96	154.62	139.33	128.51	120.62	114.72	110.22	106.71	103.93	101.69
3.212	230.40	193.30	168.94	152.30	140.53	131.96	125.57	120.69	116.89	113.89	111.48
3.376	250.39	210.09	183.67	165.63	152.89	143.63	136.72	131.45	127.36	124.13	121.54

Table 2: Difference of the deflections from the present study and that of [2] for given values of the load parameter

$\frac{\sigma_x a^2}{Et^2}$	w/t		
	Present	Samuel Levy	% diff
3.66	0	0	0
3.72	0.195	0.25	22.00
3.96	0.422	0.498	15.26
4.34	0.632	0.743	14.94
4.87	0.842	0.984	14.43
5.51	1.041	1.22	14.67
6.3	1.243	1.45	14.28
7.22	1.443	1.673	13.75
8.24	1.636	1.889	13.39
9.38	1.828	2.101	12.99
10.61	2.015	2.303	12.51
11.99	2.206	2.498	11.69
13.48	2.395	2.687	10.87
14.97	2.57	2.871	10.48
16.79	2.769	3.044	9.03
18.77	2.97	3.212	7.53
21.45	3.223	3.376	4.53

Table 3: Stress parameter for given values of deflection using a trigonometric function

w/t	$\frac{\sigma_x a^2}{Et^2}$		
	Present	Samuel Levy	% diff
0	3.65	3.66	0.27
0.25	3.76	3.72	1.08
0.498	4.08	3.96	3.03
0.743	4.6	4.34	5.99
0.984	5.31	4.87	9.03
1.22	6.2	5.51	12.52
1.45	7.26	6.3	15.24
1.673	8.45	7.22	17.04
1.889	9.77	8.24	18.57
2.101	11.22	9.38	19.62
2.303	12.74	10.61	20.08
2.498	14.35	11.99	19.68
2.687	16.02	13.48	18.84
2.871	17.78	14.97	18.77
3.044	19.53	16.79	16.32
3.212	21.33	18.77	13.64
3.376	23.18	21.45	8.07

The present study also tried to find out at what deflection to thickness ratio will the post-buckling (large deflection analysis) will approximately be equal to critical buckling load (small deflection analysis). This is present in Table 4. The w/t ratios from various load parameters were computed and tabulated in the fourth column. The load parameters were converted to buckling (and post-buckling) loads and tabulated in the second column. The buckling loads were approximated to the nearest integer and tabulated in the third column. The w/t ratios that give loads approximately equal to the critical buckling load are those below the value of 0.225.

A similar thing was done in Table 5. In the Table, a set of w/t ratios from 0 to 0.4 at an increment of 0.025 were used. The values of the load parameter, buckling (and post-buckling) load, and approximate buckling load (approximated to the nearest integer) corresponding to the selected w/t were calculated and tabulated on the second, third, and fourth columns of Table 5. The observation made here is the same as that made in Table 4. The critical w/t is 0.225. Above this w/t of 0.225, the buckling load will not be the same as the critical buckling when approximated to the nearest integer.

These observations imply that as long as the w/t is less than or equal to 0.225, a small deflection (linear) analysis can be used. Above this critical w/t, one must use a large (nonlinear) deflection analysis.

Table 4: Critical w/t above which large deflection analysis must be used.

$\frac{\sigma_x a^2}{Et^2}$	$\frac{N_x a^2}{D}$	$N_x a^2/D$ (Approximate to the nearest integer)	w/t
3.66	39.53432	40	0.055
3.67	39.64234	40	0.094
3.68	39.75036	40	0.121
3.69	39.85838	40	0.143
3.7	39.96639	40	0.162
3.71	40.07441	40	0.179
3.72	40.18243	40	0.195
3.73	40.29045	40	0.209
3.74	40.39846	40	0.223
3.75	40.50648	41	0.236
3.76	40.6145	41	0.248
3.77	40.72251	41	0.259
3.78	40.83053	41	0.270
3.79	40.93855	41	0.281
3.8	41.04657	41	0.291
3.81	41.15458	41	0.301
3.82	41.2626	41	0.311

IV. CONCLUSION

The present work has formulated a general buckling/post-buckling equation for rectangular thin plate analysis. The approach used here has circumvented the use of von Karman nonlinear equations and the Airy's functions. The results of this new approach when compared with available literature agree closely as discussed above. It is shown that post-buckling load or stress decreases with an increase in aspect ratio and increases with an increase in w/t. Also, it is observed that above the value of w/t = 0.225 nonlinear analysis has to be carried out since small deflection theory no longer applies. We, therefore, recommend this new equation for the analysis of thin rectangular isotropic plates as a better and general alternative to the popular von Karman large deflection equation.

Table 5: Critical w/t that gives the critical buckling load from small deflection analysis.

w/t	$\frac{\sigma_x a^2}{Et^2}$	$\frac{N_x a^2}{D}$	$N_x a^2 /D$ (Approximate to the nearest integer)
0	3.655	39.48032	39
0.025	3.656	39.49112	39
0.05	3.659	39.52352	40
0.075	3.664	39.57753	40
0.1	3.672	39.66395	40
0.125	3.682	39.77196	40
0.15	3.693	39.89078	40
0.175	3.707	40.04201	40
0.2	3.723	40.21483	40
0.225	3.742	40.42007	40
0.25	3.762	40.6361	41
0.275	3.784	40.87374	41
0.3	3.809	41.14378	41
0.325	3.836	41.43543	41
0.35	3.865	41.74868	42
0.375	3.896	42.08353	42
0.4	3.929	42.43999	42

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APPENDIX

$$k_{bT} = k_{bx} + \frac{2k_{bxy}}{\beta^2} + \frac{k_{by}}{\beta^4};\ k_{mT} = k_{mx} + \frac{2k_{mxy}}{\beta^2} + \frac{k_{my}}{\beta^4}$$

$$k_{bx} = \iint \left(\frac{d^2h}{dR^2}\right)^2 dR\, dQ = \int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR \times \int_0^1 {h_y}^2\, dQ;$$

$$k_{bxy} = \iint \left(\frac{d^2h}{dR\, dQ}\right)^2 dR\, dQ = \int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR\ \times \int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ$$

$$k_{by} = \iint \left(\frac{d^2h}{dQ^2}\right)^2 dR\, dQ = \int_0^1 {h_x}^2\, dR \times \int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ$$

$$k_{mx} = \iint \left(\frac{\partial h}{\partial R}\right)^4 dR\, dQ = \int_0^1 \left(\frac{dh_x}{dR}\right)^4 dR \times \int_0^1 {h_y}^4\, dQ$$

$$k_{mxy} = \iint \left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 dR\, dQ = \int_0^1 \left(\frac{dh_x}{dR}\right)^2 {h_x}^2\, dR\ \times \int_0^1 \left(\frac{dh_y}{dQ}\right)^2 {h_y}^2 dQ$$

$$k_{my} = \iint \left(\frac{\partial h}{\partial Q}\right)^4 dR\, dQ = \int_0^1 {h_x}^4\, dR \times \int_0^1 \left(\frac{dh_y}{dQ}\right)^4 dQ$$

$$k_{Nx} = \iint \left(\frac{dh}{dR}\right)^2 dR\, dQ = \int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR\ \times\ \int_0^1 {h_y}^2\, dQ$$

For ss strip

$$h = \operatorname{Sin}\pi R\,.\operatorname{Sin}\pi Q = h_x.h_y; \qquad for\ m = n = 1$$

$$h_x = \operatorname{Sin}\pi R\,;\ \frac{dh_x}{dR} = \pi\, Cos\, \pi R; \frac{d^2h_x}{dR^2} = -\pi^2 \operatorname{Sin}\pi R$$

$$(h_x)^2 = \operatorname{Sin}^2\pi R\,;\ \left(\frac{dh_x}{dR}\right)^2 = \pi^2 Cos^2\pi R; \left(\frac{d^2h_x}{dR^2}\right)^2 = \pi^4 \operatorname{Sin}^2\pi R$$

$$(h_x)^4 = \operatorname{Sin}^4\pi R\,;\ \left(\frac{dh_x}{dR}\right)^4 = \pi^4 Cos^4\pi R; \left(\frac{dh}{dR}\right)^2 = \left(\frac{dh_x}{dR}\right)^2.\ (h_y)^2 = \pi^2 Cos^2\pi R. \operatorname{Sin}^2\pi Q$$

$${h_x}^2.\left(\frac{dh_y}{dQ}\right)^2 = \operatorname{Sin}^2\pi R\,.\pi^2 Cos^2\pi Q$$

$$\int_0^1 Sin\ \pi R\, dR = \left[-\frac{\cos\pi R}{\pi}\right]_0^1 = -\frac{1}{\pi}[\cos\pi - \cos 0] = \frac{2}{\pi}$$

$$\int_0^1 Sin^4\ \pi R\, dR = \frac{3}{8};\ \int_0^1 Cos^4\ \pi R = \frac{3}{8};\ \int_0^1 Sin^2\ \pi R = 0.5;$$

$$\int_0^1 Cos^2\ \pi R = 0.5;\ \int_0^1 Sin^2\ \pi R\,.Cos^2\ \pi R\, dR = \frac{1}{8}$$

$$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR = \int_0^1 (\pi^4 \operatorname{Sin}^2\pi R)\, dR = \pi^4 \times 0.5 = 0.5\pi^4$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR = \int_0^1 (\pi^2 Cos^2\pi R)\, dR = \pi^2 \times 0.5 = 0.5\pi^2$$

$$\int_0^1 {h_x}^2\, dR = \int_0^1 (\operatorname{Sin}^2\pi R)\, dR = 0.5$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^4 dR = \int_0^1 (\pi^4 Cos^4\pi R)\, dR = \pi^4 \times \frac{3}{8} = \frac{3}{8}\pi^4$$

$$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 {h_x}^2\, dR = \int_0^1 (\pi^2 Cos^2\pi R\,.\mathrm{Sin}^2\,\pi R)\;dR = \pi^2 \times \frac{1}{8} = \frac{1}{8}\pi^2$$
$$\int_0^1 {h_x}^4\, dR = \int_0^1 \mathrm{Sin}^4\,\pi R\, dR = \frac{3}{8}$$

$$\int_0^1 {h_y}^2\, dQ \int_0^1 \mathrm{Sin}^4\,\pi R\, dR = \frac{3}{8}$$

$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ = \int_0^1 (\pi^2 Cos^2\pi R)\, dR = \pi^2 \times 0.5 = 0.5\pi^2$$
$$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ = \int_0^1 (\pi^4\,\mathrm{Sin}^2\,\pi R)\, dR = \pi^4 \times 0.5 = 0.5\pi^4$$
$$\int_0^1 {h_y}^4\, dQ \;= \int_0^1 \mathrm{Sin}^4\,\pi R\, dR = \frac{3}{8}$$
$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 {h_y}^2 = \int_0^1 (\pi^2 Cos^2\pi R\,.\mathrm{Sin}^2\,\pi R)\;dR = \pi^2 \times \frac{1}{8} = \frac{1}{8}\pi^2$$
$$\int_0^1 \left(\frac{dh_y}{dQ}\right)^4 dQ = \int_0^1 (\pi^4 Cos^4\pi R)\, dR = \;\pi^4 \times \frac{3}{8} = \frac{3}{8}\pi^4$$

$$k_{Nx} = \iint \left(\frac{dh}{dR}\right)^2 dR\, dQ = \int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR \;\times\; \int_0^1 {h_y}^2\, dQ$$

For Plates

S/N	SSSS	X part of Stiffness coefficient ss		Y part of Stiffness coefficient		Stiffness coefficient
1	k_{bx}	$\int_0^1 \left(\frac{d^2h_x}{dR^2}\right)^2 dR$	$0.5\pi^4$	$\int_0^1 {h_y}^2\, dQ$	0.5	$0.25\pi^4$
2	k_{bxy}	$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR$	$0.5\pi^2$	$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 dQ$	$0.5\pi^2$	$0.25\pi^4$
3	k_{by}	$\int_0^1 {h_x}^2\, dR$	0.5	$\int_0^1 \left(\frac{d^2h_y}{dQ^2}\right)^2 dQ$	$0.5\pi^4$	$0.25\pi^4$
4	k_{mx}	$\int_0^1 \left(\frac{dh_x}{dR}\right)^4 dR$	$\frac{3}{8}\pi^4$	$\int_0^1 {h_y}^4\, dQ$	$\frac{3}{8}$	$\frac{9}{64}\pi^4$
5	k_{mxy}	$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 {h_x}^2\, dR$	$\frac{1}{8}\pi^2$	$\int_0^1 \left(\frac{dh_y}{dQ}\right)^2 {h_y}^2 dQ$	$\frac{1}{8}\pi^2$	$\frac{1}{64}\pi^4$
6	k_{my}	$\int_0^1 {h_x}^4\, dR$	$\frac{3}{8}$	$\int_0^1 \left(\frac{dh_y}{dQ}\right)^4 dQ$	$\frac{3}{8}\pi^4$	$\frac{9}{64}\pi^4$
7	k_{Nx}	$\int_0^1 \left(\frac{dh_x}{dR}\right)^2 dR$	$0.5\pi^2$	$\int_0^1 {h_y}^2\, dQ$	0.5	$0.25\pi^2$