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Original Article

Refined Mixed Finite Element Model For Analysis of Bar Member

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Abstract *- The mixed finite element method allows the stress field to be the primary and displacement variable. The variation of these primary variables is selected in the problem domain and then solved with the minimization of the total potential energy. Unlike other mixed finite element formulation which uses mixed variation principle like Hellinger-Reissner's variational principle, which is based on the stationarity principle, the present formulation is based on the minimization principle; hence it gives a stable algorithm. The developed formulation is applied to solve the cantilever bar member subjected to varying axial load. The result was compared with the closed-form analytical solution and displacement-based finite element formulation. The present formulation gives accurate results for the stress value with only one element compared to the three-element based on displacement-based formulation.*

Keywords *- Mixed-finite element, Minimization of total potential energy principle, Bar element, Refined solution, HR principle.*

1. Introduction

Determination of stress in the solid mechanic's domain is crucial. The accurate stress values are responsible for the safe design of the structure.

The partial differential equation governs all the solid mechanic's problems. These partial differential equations have a solution that describes the particular variable in the problem domain. Closed-form analytical solution is only possible for the simple loading case and simply supports geometry. This solution is available primarily for the Navier type scheme. Solving the partial differential equation analytically for a practical case is challenging. To address this approximate issue solution is used. The finite element method is the most established method for performing the numerical analysis of the partial differential equation. The finite element method is displacement-based. This means that the displacement is the degree of freedom, and approximate values of the displacement are obtained as a solution of finite element formulation. A post-processing operation is required to obtain the stress values of any problem. In this process, displacement values are used to calculate strain by using the strain-displacement relation; this strain is then used to calculate stress values using the stress-strain relation. This process leads to the accumulation of errors, leading to more differences in the stress values. These theories can be referred from the[1].

On the other hand, the mixed finite element method allows for choosing the stress or strain along with the displacement as primary variables. This method uses Hellinger Reissner's principle[2], [3], Hu-Washi-Zu, and many other mixed variational principles to obtain the solution. A mixed finite element allows the independent variation of stress and displacement field, and one can directly obtain the stress quantities after solution. This method reduces the long post-processing route, reducing the error of approximation. Although mixed finite element method also has many drawbacks. Significant drawbacks are, Firstly, this method chooses independent variation of stress and displacement quantities which violates the fundamental law of elasticity, as stress is dependent on the displacement. Secondly, the application of this method is complex and cumbersome. Thirdly, this method uses mixed variational principles, which are stationarity-based. Hence the algorithm of the solution is not stable for this formulation.

To overcome the deficiency of the mixed finite element model Desai and Ramtekkar[4], [5]developed the eighteen nodes and six nodes mixed finite element model with analyzing three and two-dimensional problems. These formulations are based on the minimization of the total potential energy principle. This formulation contains the transverse stresses and displacement quantities as primary variables. Variations of the stress fields are invoked with the variation of the displacement field by applying the fundamental elasticity relations (stress-strain relation and strain-displacement relation). Hence, the mixed finite element model developed by Desai and Ramtekkar[4], [5]has removed all the drawbacks of mixed finite element and displacement-based formulation. Although the mixed formulation developed by Desai and Ramtekkar[4], [5]did not satisfy the equilibrium equation in the model, they still satisfy the equilibrium equation in the approximate sense. They applied this model to the fiber-reinforced composite plates and beamed[6]–[11], and obtained the results compared to an analytical closed-form solution.

Present formulation refines Desai and Ramtekkar's[4], [5] mixed finite element formulation by including the equilibrium equation. This methodology is previously used in the two-dimensional formulation by Patel and Ramtekkar[12]. However, this formulation is the first time used in the one-dimensional formulation. Although it might appear complex for the application in an isotropic material, applying this formulation in the interlaminar stress variation of laminated composite is beneficial, as discussed by[13]. Section 2 of the work gives the theoretical formulation of the refined mixed finite element formulation, and in section 3, the numerical application of the developed formulation is shown. Section 4 of the paper discusses the formulation's result and advantages and disadvantages. Finally, section 5 concludes the work and discusses the future scope of the work.

2. Theoretical Formulation

The domain of the formulation is considered in the x-direction only. Consider the bar lying in the x-direction $0 < x < l$. This is shown in Figure 1. The discretization details of the problem are shown in Figure 2.

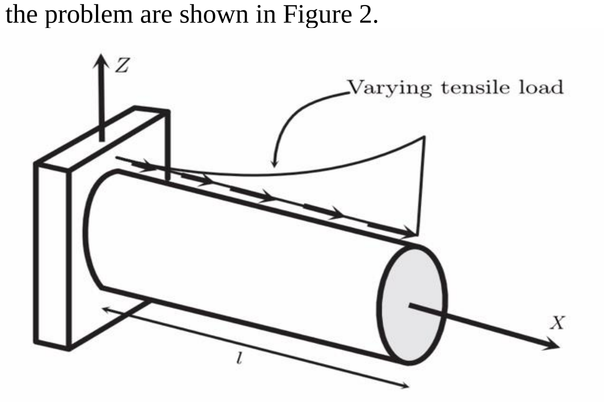

Fig. 1 Details of the axial bar subjected to varying tensile load.

Selection of approximation function to predict the variation of the displacement and stress primary variables in the formulation. As this formulation incorporates the equilibrium equation, terms in the approximation function must also adopt that.

$$\boldsymbol{u(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5} \quad (1)$$

Equation 1 shows the approximate variation of the primary displacement variable in the domain of the problem. Elasticity relation has been used to invoke the primary stress variable from the above function. Stress values can be calculated from the strain variation. Strain and displacement are related by the kinematic equation, as shown in Equation 2.

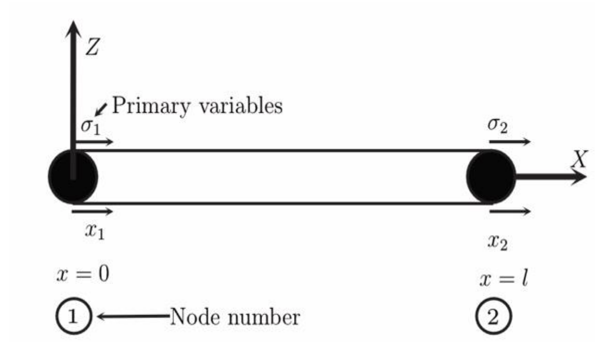

Fig. 2 Element level discretization of the one-dimensional bar element

$$\varepsilon = \frac{du}{dx} \quad (2)$$

To obtain the strain variation according to Equation 2, displacement variation needs to be differentiated concerning$x$. Hence, from Equation 1 and Equation 2:

$$\varepsilon(x) = a_1 + 2a_2 x \ \ +3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \quad (3)$$

Equation 3 shows the variation of the strain in the domain. Strain and stress are related by constitute relation, as shown in Equation 4.

$$\sigma_x = E \times \varepsilon \quad (4)$$

Substituting the strain variation function as shown in Equation 3 into Equation 4. Stress variation in the domain is shown in Equation 5.

$$\sigma_x(x) = E(a_1 + 2a_2 x \ \ +3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4) \quad (5)$$

To incorporate the equilibrium equation shown in Equation 6, stress variation has to be differentiated concerning$x$, as shown in Equation 7.

$$-p_x = \frac{d\sigma_x}{dx} \quad \text{as} \quad (6)$$

Equation 7 shows the variation of the body force in the domain.

$$-p_x(x) = E(2a_2 x \ \ +6a_3 x \ \ +12a_4 x^2 + 20a_5 x^3) \quad (7)$$

Using Equation 1, Equation 5, and Equation 7 and putting the $x = 0, x = l$ in the variation, the nodal variation of the primary variable can be obtained. This variation is shown in Equation 8 to Equation 13.

$$\boldsymbol{u_1 = a_0} \quad (8)$$

$$\boldsymbol{u_2 = a_0 + a_1 l + a_2 l^2 + a_3 l^3 + a_4 l^4 + a_5 l^5} \quad (9)$$

$$\boldsymbol{\sigma_1} = \mathbf{E} \tag{10}$$

$$\sigma_2 = E(a_1 + 2a_2 l \ \ +3a_3 l^2 + 4a_4 l^3 + 5a_5 l^4) \tag{11}$$

$$-p_1 = 2Ea_2 \tag{12}$$

$$-p_2(x) = E\left(2a_2 l \ \ +6a_3 l \ \ +12a_4 l^2 + 20a_5 l^3\right) \tag{13}$$

Equation 14 shows the matrix relation after arranging all the equations in matrix form and rearranging it for body forces.

$$\begin{Bmatrix} u_1 \\ \sigma_1 \\ u_2 \\ \sigma_2 \\ -p_1 \\ -p_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & 0 & 0 \\ 1 & l & l^2 & l^3 & l^4 & l^5 \\ 0 & E & 2El & l^2 & l^3 & 5El^4 \\ 0 & 0 & -2Ea & 0 & 0 & 0 \\ 0 & 0 & -2Ea & -6Eal & -12Eal^2 & -20Eal^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{Bmatrix} \tag{14}$$

This relation can be physically interpreted. It is shown in Equation 15.Where $\{d\ \}$ is the vector containing the primary variables, $\{\alpha\}$ is the constant vector and $[A]$Is the influence vector contains the variable values?

$$\{d\} = [A]\{\alpha\} \tag{15}$$

Equation 1 can be represented in the matrix where $\{U\}$ is the global displacement vector and $[X]$Is the variable matrix. The value of $\{\alpha\}$Vector can be substituted from Equation 15 to Equation 16.

$$\{U\} = [X]\{\alpha\} \tag{16}$$

After substituting the value of $\{\alpha\}$ global displacement vector $\{U\}$ can be related tothe nodal displacement vector $\{d\}$.

$$\{U\} = [X][A]^{-1}\{d\} \tag{17}$$

The term related to the global displacement vector $\{U\}$ with the nodal displacement vector $\{d\}$ is called interpolation matrix. Hence the interpolation matrix $[N]$ can be evaluated with the $[X]$ and $[A]$The matrix is shown in Equation 18 and Equation 19.

$$\{U\} = [N]\{d\} \tag{18}$$

$$[N] = [X][A]^{-1} \tag{19}$$

When interpolation matrix is multiplied with Jacobean matrix strain matrix $[B]$Can be obtained as shown in Equation 20.

$$[B] = [L][X][A]^{-1} \tag{20}$$

after getting the strain matrix and applying the minimum total potential energy principleproperty matrix for an element can be calculated as shown in Equation 21.

$$K^e = \int_V [B]^T[E][B]dv = A\int_0^l [B]^T[E][B]dx \tag{21}$$

For the illustrative variation of the axial load, the nodal force vector can be expressed as shown in Equation 22.

$$f^e = \int_0^l [N]^T P(x)dx \tag{22}$$

After combining Equation 21 and Equation 22 and separating the contribution from the body force and primary variables, as shown in equation 23.

$$\begin{bmatrix} [K]_{pp} & [K]_{pb} \\ [K]_{bp} & [K]_{bb} \end{bmatrix} \begin{Bmatrix} \{d\}_p \\ \{d\}_b \end{Bmatrix} = \begin{Bmatrix} \{f\}_p \\ \{f\}_b \end{Bmatrix} \tag{23}$$

A static condensation procedure has been performed to eliminate the body force term from Equation 23. After static condensation, the new relationship has been obtained, as shown in Equation 24. This is the refined mixed finite element relation.

$$[K]^{e*}\{d\}^e = \{f\}^{e*} \tag{24}$$

Here, $[K]^{e*}$ is the element level refined property matrix, and superscript is$\{f\}^{e*}$ is the element level influence vector. Soling the equation shown in Equation 24 gives the refined mixed finite element solution.

3. Numerical Application

Validation of the above developed refined mixed finite element model bar fixed at one and free at another end is considered. This bar is subjected to varying tensile load in the direction. The material property of the bar is shown in Table 1. Table 2 shows the boundary condition of the problem. The MATLAB code is written for bar analysis by the refined mixed finite element method and simple displacement-based finite element method. The closed-form analytical solution based on elasticity theory is considered the reference for the case. The bar is discretized for the refined mixed finite element and displacement-based finite element model. 1 element is considered for the refined mixed finite element model, and 1 and 3 elements are considered for the displacement-based finite element model. Gauss quadrature technique is used to perform the numerical integration of both models.

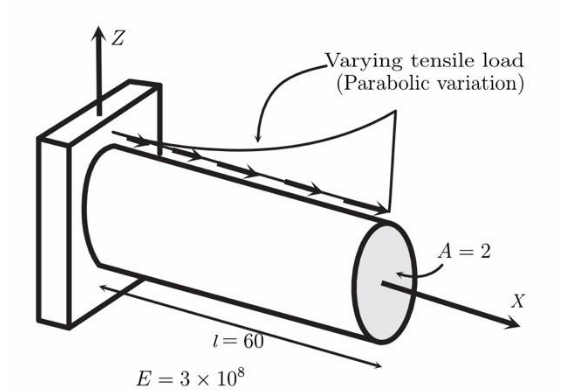

Fig. 3 One dimensional bar element subjected to varying tensile force fixed from one end

Table 1. Material property of the Bar

S.No.	Property	Value (No Unit assigned)
1.	Elastic Modulus (E)	$3\text{x}10^8$
2.	Area (A)	2
3.	Length (L)	60

Table 2. Boundary condition of the problem

S.No.	Primary variable	Values	
		$x = 0$	$x = l$
1.	u	0	-
3.	σ	-	0

4. Results and Discussion

Bar subjected to varying axial load is analyzed with the refined mixed finite element formulation and displacement-based finite element formulation. The value of displacement is obtained almost equal in all four cases. However, stress values show considerable variation. The stress value for single element refined mixed finite element formulation is the same as analytical closed-form solution. The stress value for the single and three-element displacement-based finite element shows significant deviation from the analytical closed-form solution, as shown in Table 3. The percentage difference in the single and three-element finite element model is 25% and 12.5%, respectively.

Table 3. Results obtained after analysis

PV[b]	Refined Analysis (1)[a]	Analytical (Closed form)	FEM (1) [a]	FEM (3) [a]
u_1	0	0	0	0
u_1	6000	6000	4500	5250
σ_1	0.009	0.008	0.009	0.0089
σ_2	0	0	0	0

[a](numbers denote the number of elements used in that numerical scheme); [b](Primary Variables)

5. Conclusion

A refined mixed finite element model has been developed in the present work. This two-node model has stress and displacement as the primary variables. In addition, this model uses an equilibrium equation to refine the formulation. As a result, the present model satisfies all three-fundamental elasticity relations and explicitly satisfies the nodes' equilibrium equation.

Although this formulation appears uneconomical for the isotropic case and simple bar problem, the level of accuracy achieved is highly appreciable. The application of this formulation may be made in the laminated composite structure where delamination may occur due to inter-laminar transverse stress variation in the thickness direction.

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