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Review Article

Review of Mixed Finite Element Approach for the Analysis of Fiber Reinforced Polymer Composite Laminates

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Abstract - *Composite laminates are very popular nowadays. With the advancement of machines and quality, there is a shift in the use pattern of composite materials. As its importance increases, the requirement for precise analysis also increases. Various methods have been developed to analyze the inter-laminar stresses in the laminated composites accurately. In this work critical review of the approach made to analyze the laminated composite beam, with a particular focus on the mixed finite element method, has been done*.

Keywords - *Mixed finite element method, Composite laminate beam, Rissner's mixed variational theorem, Higher-order shear deformation theory.*

1. Introduction

The composite materials are prevalent because of their high strength-to-weight ratio. It is now being used as a prominent member; hence the requirement for precise analysis also increases. The laminated composite's inter-laminar stresses are the leading cause of failure analysis. In general, it varies in the normal range, but near the free edge of cuts, these transverse stresses shoot up, which leads to delamination failure. Hence, precise analysis to determine the stress values is required.

Different theories to model the behaviour of laminated composite beams have been developed. Initially, researchers were modelling it as equivalent single layer theory[1]–[4], in which they model the laminate as a single layer to predict the displacement behaviour. This philosophy is good for isotropic beams and the laminated beam's global behaviour. However, ESL theory cannot predict the variation of inter-laminar transverse stresses at the ply level. *Zig-zag theories* are game-changing approaches that predict the displacement profile's kink by adding extra function. The review of zig-zag theory can be found in the review paper by Carrera[5] apart from various sources [6]–[16],which worked on the analysis of composite beams using zig-zag theory. The mixed theory is the most accurate and versatile so far. The only problem with mixed theories is that they are complex to apply. RMVT gave a new dimension to the mixed model, and various models have been developed like[17]–[21]. In his work [22],Carrera discusses the different ideas for developinga mixed finite element model. Miguel and Carrera [21] performed the accurate stress analysis using one dimensional mixed finite element model. Ramtekkar and Patel [23], [24]recently developed the refined mixed finite element model and discussed the accuracy of the computational economy of the formulation.

This paper reviews the mixed finite element model to analyze the laminated composite beams. Starting from Hellinger[25] in 1914, the review has been done to date. First, the Desai and Ramtekkar model has been discussed in detail. The refined mixed finite element model developed by Ramtekkar and Patel has also been discussed. Finally, the Tonti diagram for the Desai and Ramtekkar[27] and Ramtekkar and Patel model has presented a detailed understanding of the model.

2. Mixed Finite Element Method

2.1. Definition

A multifield problem is a formulation that contains more than one primary variable. That means there may be a stress or strain field along with the displacement field. The multifield formulation can be either mixed or hybrid. A mixed formulation is if all the variables belong to the inner field.

2.2. Application of Mixed Finite Element Method in Composite Laminates

Multi-field principles were first introduced in the area of solid mechanics by Hellinger[25]. He presented a finite element model for a finite elasticity problem in which stress and displacement were varied independently. For the linear elasticity problem,Prange modified Hellinger's principle by

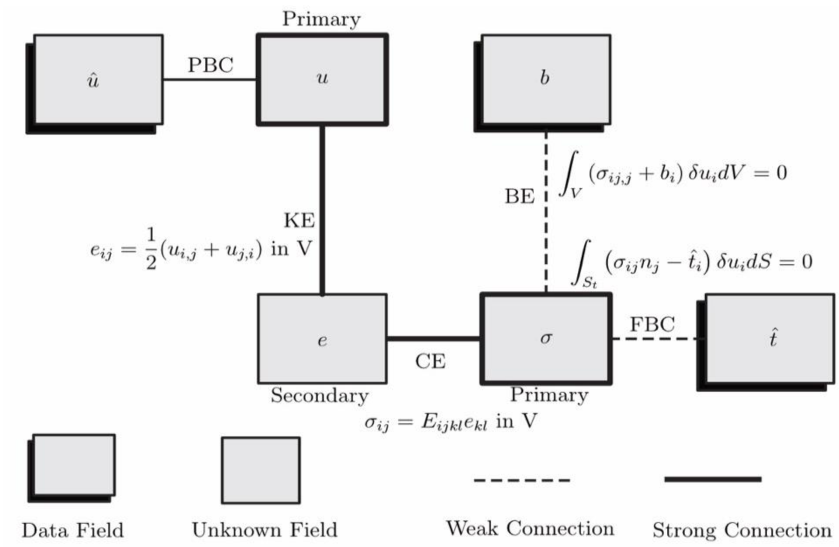

Fig. 1 Tonti diagram for Desai and Ramtekkar [18] mixed finite element model

including the boundary condition. Reissner[26] unaware of Hellinger's work, presented a variational theorem for linear elasticity with independent displacement and stress fields, which includes the boundary condition.

In Hellinger and Reissner theorem they varied stress and displacement independently. They were allowed to select any stress out of six stress quantities. Later on, for the laminated composite Reissner gave the mixed variational theorem. He argued that only those stress components can be taken as primary variables which are continuous in the thickness direction. Transverse shear stress and transverse normal stress have this property. Hence, in Reissner's Mixed Variational Theorem (RMVT) the primary variables are three displacement components and three transverse stresses.

2.3. Mixed finite element formulation by Desai and Ramtekkar[27]

Desai and Ramtekkar[27]2 developed a six-node mixed finite element method which was used to analyze the composite laminate beams and plates in their consecutive work [28]–[34]. In this section detailed review of their work is done. The core idea is to develop a method that can have the beauty of the mixed finite element method and the minimization principle's stability. To achieve this goal, they considered transverse stress as the primary variable along with the displacements. The variation of these primary variables is not independent like other mixed finite element methods. They invoked the displacement variation from another mixed finite element method by using the fundamental elastic relation (stress-strain relation and strain-displacement relation). In this process, they satisfied the two-fundamental elastic relation explicitly. This formulation then uses the minimum total potential energy principle to obtain the approximate solution. The use of the minimum potential energy principle provides stability to the formulation. This formulation gives very accurate results as compared to the analytical solution.

Fig. 1 shows the Tonti diagram for the Desai and Ramtekkar mixed finite element model. This figure shows the data field, unknown field, and weak and stronglinks. The data field box shows the external data like the nodal displacement value presented by $\hat{\boldsymbol{u}}$ and traction values $\hat{\boldsymbol{t}}$And the body force, if any. Primary written near the box shows the primary variables like in the present formulation displacement and transverse stress are the primary variables. Secondary variables are those whose direct variation is not considered in the formulation. Like the strain isnot considered as a primary variable in the formulation. Nodal displacement values presented by tare bold $\hat{\boldsymbol{u}}$ is connected with displacement primary variable by primary boundary condition (PBC). This is a strong link as the primary boundary condition is the essential boundary condition in the formulation. '$\boldsymbol{b}$' denote the body force in the diagram and that$\hat{\boldsymbol{t}}$ denotes the traction values that$\hat{\boldsymbol{t}}$ is connected with the stress primary variables by force boundary condition,which is the weak link. Another weak link, in theory, is the $\boldsymbol{\sigma}$ and $\boldsymbol{b}$. The connection between them is through the balance equation (equilibrium equation). The connection between the $\boldsymbol{u}$ and $\boldsymbol{\varepsilon}$ is that strong connection, and same the for the connection between $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ as the $\boldsymbol{\sigma}$ is also a master field in the formulation. The variation of stress has invoked the displacement, and hence the link ($\rightarrow \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\sigma}$) has to be strong. The only variation option reaming in the model is the balanced equation.

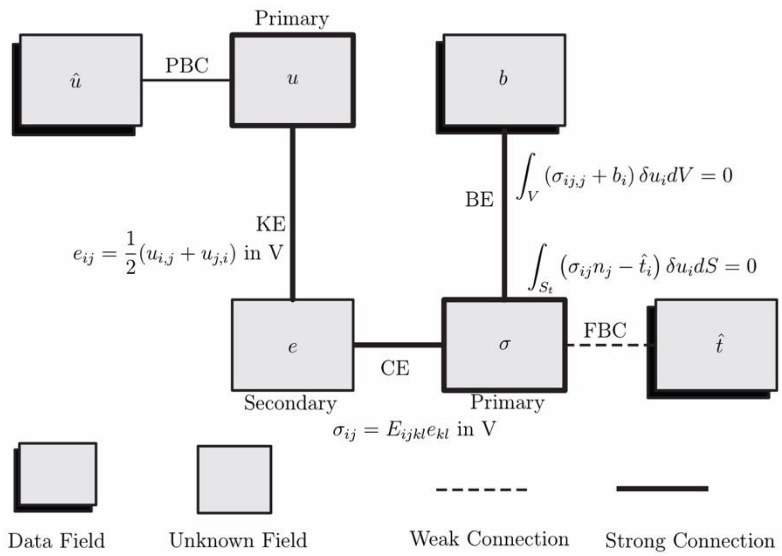

Fig. 2 Tonti diagram for Ramtekkar and Patel refined mixed finite element model

2.4. Refined **m***ixed finite element formulation by Ramtekkar and Patel*

Ramtekkar and Patel's[23] refined mixed model is an improvement over the Desai and Ramtekkar model. In the earlier model, they did not consider the equilibrium equation, which has been incorporated in the new formulation. Ramtekkar and Patel's model considers the equilibrium equation through the body forces and applies static condensation to refine the variation of primary variables. Applying this step before minimization satisfies the equilibrium equation explicitly at the nodes. This step gives extra refinement to the model and makes the model economical as compared to the Desai and Ramtekkar model.

Fig. 2 shows the Tonti diagram for the refined mixed finite element model. In this diagram, the link between the body force and stress also becomes strong; hence the equilibrium equation is satisfied explicitly at the nodes. The only weak link is in the force boundary condition. This improvement in the model increases the efficiency and reduces the computational cost compared to Desai and Ramtekkar's models.

3. Conclusion

Analysis of composite laminates is a crucial task. Interlaminar stresses cause delamination failure. Hence, the 3D state of stress is required for the precise and confident analysis of the laminated composite beams. Development of theories to analyze the laminated composite beam has been presented in the timeline fashion in the present work. A significant contribution has been discussed in great detail inwork. Various models based on the RMVT principle have also been discussed in the present work.

The mixed finite element model developed by Desai and Ramtekkar [27] significantly contributes significantly to stress determination at the ply level. This formulation is versatile as it is finite element based. This enables this formulation to model complicated geometry as well. Considering stress as a primary variable removes the error due to post-processing. This formulation is a significant landmark achievement in laminated composite analysis.

Tonti diagram for the Desai and Ramtekkar model and refined Ramtekkar and Patel model have also been presented inwork. Tonti diagram help to understand clearly the variation and the weak link where the approximation is being performed.

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