Optimal control for a distributed parameter system with delayed-time. Application to one-sided heat conduction system

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Abstract: *This paper gives a solution of an optimal control problem for a distributed parameter system (DPS) with delayed-time, governed by a heat-conduction equation, using the numerical method. In which, the delayed object $e^{-\tau s}$ is replaced by using first-order Pade approximation model. The system is applied to a specific one-sided heat-conduction system in a heating furnace to control temperature for the objects which have flat-slab shape following the most accurate burning standards [2], [6]. The aim of problem is to find an optimal control signal (optimal voltage) so that the error between the distribution of real temperature of the object and the desired temperature is minimum after a given period of time t_f [2], [6], [9]. To verify the solution of the problem, we have proceeded to run the simulation programs on a flat-slab of Carbon steel and a flat-slab of Samot.*

Keywords: *optimal control, distributed parameter systems, delay,numerical method, Pade approximation*

I. INTRODUCTION

Optimal control for distributed parameter system is applied in many fields such as heat treatments, composting the magnetic materials, steel rolling, etc.

In some previous technologies [2], [6], [7], the heating process was carried out in a burning furnace with FO heavy oil, such as burning in steel rolling or in the processes of manufacture of aluminum, glass. In this case, the transfer function of the furnace is the delayed inertia, and the relationship between the temperature of furnace is the parabolic-type partial differential equation with the boundary condition of type 3. If we consider the optimal control problem for the "most accurate burning process", the control object is now distributed parameter system with delayed-time

II. THE PROBLEM OF OPTIMAL CONTROL

A. The object model

As a typical distributed parameter system, the one-sided heat conduction system is considered. The process of one-sided heating of the objects which have flat-slab shape in a furnace is described by the parabolic-type partial differential equation, as follows in [2], [6], [8], [9], [10]:

$$a\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t} \tag{1}$$

where $q(x,t)$, the temperature distribution in the object, is the output needing to be controlled, depending on the spatial coordinate x with $0 \le x \le L$ and the time t with $0 \le t \le t_f$, a is the temperature-conducting factor (m^2/s), L is the thickness of object (m), t_f is the allowed burning time (s)

The initial and boundary conditions are given in [2],[6],[9],[10].

$$q(x,0) = q_0(x) = const \tag{2}$$

$$\lambda\left.\frac{\partial q(x,t)}{\partial x}\right|_{x=0} = \alpha\left[q(0,t) - v(t)\right] \tag{3}$$

$$\left.\frac{\partial q(x,t)}{\partial x}\right|_{x=L} = 0 \tag{4}$$

with α as the heat-transfer coefficient between the furnace space and the object (W/m^2.^{0}C), λ as the heat-conducting coefficient of material (W/m.^{0}C), and $v(t)$ as the temperature of the furnace respectively (^{0}C).

The temperature $v(t)$ of the furnace is controlled by voltage $u(t)$, the temperature distribution $q(x,t)$ in the object is controlled by means of the fuel flow $v(t)$, this temperature is controlled by voltage $u(t)$. Therefore, the temperature distribution $q(x,t)$ will depend on voltage $u(t)$.

The relationship between the provided voltage for the furnace $u(t)$ and the temperature of the furnace $v(t)$ is ussually the first order inertia system with delayed-time as in [1], [2], [6], [9],[10].

$$T.\dot{v}(t) + v(t) = k.u(t-\tau) \tag{5}$$

where T is the time constant, τ is the time delay; k is the static transfer coefficient; $v(t)$ is the temperature of the furnace and $u(t)$ is the provided voltage for the furnace (controlled function of the system).

B. The objective function and the constrained conditions

In this case, the problem is set out as follows: we have to determine a control function $u(t)$ with ($0 \le t \le t_f$) so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It means at the end of the heating process to ensure temperature uniformity throughout the whole material:

$$J[u(t)] = \int_0^L \left[q^*(x) - q(x,t_f)\right]^2 dx \to \min \tag{6}$$

The constrained conditions of the control function is:

$$U_1 \le u(t) \le U_2 \tag{7}$$

with U_1, U_2 are the lower and upper limit of the supply voltage respectively (V). This problem is called the most accurate burning problem.

III. THE SOLUTION OF PROBLEM

The process of finding the optimal solution includes 2 steps:

- Step 1: Find the relationship between $q(x,t)$ and the control signal $u(t)$. Namely, we have to solve the equation of heat transfer (relationship between $v(t)$ and $q(x,t)$) with boundary condition type-3 combined with ordinary differential equation with time delay (relationship between $u(t)$ and $v(t)$)

- Step 2: Find the optimal control signal $u^*(t)$ by substituting $q(x,t)$ found in the first step into the function (6), after that finding optimal solution $u^*(t)$.

A. *Find the relationship between q(x,t) and the control signal u(t)*

To solve the partial differential equation (1) with the initial and the boundary conditions (2), (3), (4), we apply the Laplace transformation method with the time parameter t. On applying the transform with respect to t, the partial differential equation is reduced to an ordinary differential equation of variable x. The general solution of the ordinary differential equation is fitted to the boundary conditions, and the final solution is obtained by the application of the inverse transformation.

Transforming Laplace (1), we obtained:

$$a\frac{\partial^2 Q(x,s)}{\partial x^2} = sQ(x,s) \tag{8}$$

where: $Q(x,s) = L\{q(x,t)\}$

After transforming the boundary conditions (3), (4), we have:

$$\lambda \frac{\partial Q(x,s)}{\partial x}\bigg|_{x=0} = \alpha\left[Q(0,s) - V(s)\right] \tag{9}$$

$$\frac{\partial Q(x,s)}{\partial x}\bigg|_{x=L} = 0 \tag{10}$$

From Eq. (5), assuming the delayed object satisfy the condition: $6 \le T/\tau < 10$ in [5], [9],[10]. To solve this problem, the first order inertia system with time delay is replaced by first-order Pade appximation (Pade 1) Transforming Laplace (5), we obtained:

$$(Ts+1)V(s) = k.U(s).e^{-\tau s} \approx k.U(s)\frac{2-\tau s}{2+\tau s} \tag{11}$$

where: $V(s) = \mathbf{L}\{v(t)\}$; $U(s) = \mathbf{L}\{u(t)\}$ (12)

The general solution of (1) is:

$$Q(x,s) = A(s).e^{\sqrt{\frac{s}{a}}.x} + B(s).e^{-\sqrt{\frac{s}{a}}.x} \tag{13}$$

where: $A(s); B(s)$ are the parameters need to be find.

After transforming, we have the function:

$$Q(x,s) = \frac{U(s).k.(2-\tau s)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)} + e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)(2+\tau s)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L} + e^{\sqrt{\frac{s}{a}}.L}\right] - \lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L} - e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \tag{14}$$

Putting

$$G(x,s) = \frac{k.(2-\tau s)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)} + e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)(2+\tau s)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L} + e^{\sqrt{\frac{s}{a}}.L}\right] - \lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L} - e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \tag{15}$$

We have: $Q(x,s) = G(x,s).U(s)$ (16)

From (16), according to the convolution theorem, the inverse transformation of (16) is given by

$$q(x,t) = g(x,t) * u(t)$$

We can write

$$q(x,t) = \int_0^t g(x,\tau).u(t-\tau)d\tau \tag{17}$$

or

$$q(x,t) = \int_0^t g(x,t-\tau).u(\tau)d\tau \tag{18}$$

where

$$g(x,t) = \mathbf{L}^{-1}\{G(x,s)\} \tag{19}$$

Therefore, if we know the function $g(x,t)$, we will be able to calculate the temperature distribution $q(x,t)$ from control function $u(t)$. To find $q(x,t)$ in (18), we need to find the function (19). Using the inverse Laplace transformation of function $G(x,s)$ we have the following result:

$$g(x,t)=\frac{k.k_0^2\left(2+\tau k_0^2\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(2-\tau k_0^2\right)\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0^2t}+$$

$$+\frac{2k.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+$$

$$+\sum_{i=2}^{\infty}\frac{2\alpha.k\left(2+\tau.\Psi_i^2\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(2-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2t} \quad (20)$$

$$k_0=1/\sqrt{T};\;\; k_1=\sqrt{2/\tau}$$

where Ψ_i is calculated from the formula:

$$\Psi_i=\phi_i\sqrt{a}/L$$

- ϕ_i is the solution of the equation:

$$\phi.tg\phi=\alpha L/\lambda=B_i$$

- Bi is the coefficient BIO of the material.
- α is the heat-transfer factor (W/m^2.^{0}C)
- λ is the heat-conducting factor of object (W/m.^{0}C)
- L is the thickness of object (m),
- a is the temperature-conducting factor (m^2/s)
- τ is the time delay of the furnace (s)
- T is the time constant of the furnace (s)
- k is the static transfer coefficients of the furnace

Conclusions:

We have solved a system of parabolic-type partial differential equation with boundary conditions of type-3 (the relationship between *v(t)* and *q(x,t)*) combined with the ordinary differential equation with time delay (the relationship between *u(t)* and *v(t)*).

Thus, if we are not interested in the optimal problem, we can calculate the temperature field in the object when knowing the supplied voltage for the furnace (The problem knows the shell to find the cores), as follows:

The relationship between the supplied voltage for the furnace u(t) and the temperature field distribution in the object q(x,t):

$$q(x,t)=g(x,t)*u(t)=\int_0^t g(x,t-\tau).u(\tau)d\tau$$

(21)

with t_f is the allowed burning time (s).

B. Find the optimal control signal $u^*(t)$ by using numerical method

To find the $u^*(t)$, we have to minimize the objective function (6), it means:

$$J[u(t)]=\int_0^L\left[q^*(x)-q(x,t_f)\right]^2dx\to\min \quad (22)$$

where

$$q(x,t_f)=\int_0^{t_f}g(x,t_f-\tau).u(\tau)d\tau \quad (23)$$

and $q^*(x)$ is the desired temperature distribution; $q(x,t_f)$ is the real temperature distribution of the object at time $t=t_f$.

As calculated in [2], [6], [9],[10] the integral numerial method is used by applying Simson formula to the right-hand side of the objective function (22). The L, the thickness of the object, is divided into n equal lengths (n is an even number).

Similarly, it is applied to the right-hand side of the equation (23). The period of time t_f is devided into m equal intervals that m is an even number, too.

Thus, the optimal control problem is here to find u^*_j in order to minimize the objective function:

$$J_c[u^*]=L\sum_{i=0}^{n}\xi_i\left[q^*(x_i)-q(x_i,t_f)\right]^2 \quad (24)$$

where ξ_i are the weights assigned to the values of integrand at the points x_i. The values of x_i and the weights ξ_i are known for each integration formula.

If the Simpson's composite formula is used, the values of x_i and ξ_i are given by in [2], [6],[10].

$$\left.\begin{array}{l} x_i=L_i/n;(i=0,1,\dots,n)\\ \xi_0=\xi_n=1/3n\\ \xi_1=\xi_3=\xi_{n-1}=4/3n\\ \xi_2=\xi_4=\xi_{n-2}=2/3n\end{array}\right\} n \text{ is an even number}$$

Therefore, $q(x_i,t_f)$ is calculated:

$$q(x_i,t_f)\cong t_f\sum_{j=0}^{m}\xi_j g(x_i,t-\tau_j).u(\tau_j) \quad (25)$$

the values of τ_j and ξ_j are given by in [2], [6],[10].

$$\left.\begin{array}{l}\tau_j=jt_f/m\\ \xi_0=\xi_m=1/3m\\ \xi_1=\xi_3=\xi_{m-1}=4/3m\\ \xi_2=\xi_4=\xi_{m-2}=2/3m\end{array}\right\} \quad (j=0,1,2,\dots,m)$$

Putting

$$c_{ij}=t_f.\xi_j.g(x_i,t-\tau_j);\; u(\tau_j)=u_j;\; q^*(x_i)=q_i^* \quad (26)$$

Substituting (25) and (26) into (24), we obtained:

$$J[u^*]=L\sum_{i=0}^{n}\xi_i\left[q_i^*-\sum_{j=0}^{m}c_{ij}.u_j\right]^2 \quad (27)$$

The constrained conditions of the control function (Limit of the supplied voltage for the furnace) are described as follows:

$$U_1\le u_j\le U_2 \qquad (j = 0\div m)$$

(28)

The performance index (27) is a quadratic function of the variables u_j with constraints (28) are linear, the problem becomes a quadratic programming problem. This problem can be obtained by using numerical method after a finite number of iterations of computation.

Although a solution of the quadratic programming problem is obtained after a finite number of iterations of computation, but its algorithm is more complicated than that of the simplex method for linear programming. If the performance index is taken as

$$\mathrm{J} = \int_0^L \left| q^*(x) - q(x, \mathrm{t}_f) \right| dx \qquad (29)$$

instead of (22), the linear programming technique can be used directly. On applying the same procedure as mentioned above, the approximate performance index corresponding to (29) is written as

$$\mathrm{J} \cong \bar{J} = L \sum_{i=0}^{n} \xi_i \left| q_i^* - \sum_{j=0}^{m} c_{ij}.u_j \right| \qquad (30)$$

The problem of minimizing (30) under the constraints (28) can be put into a linear programming form by using known techniques [2],[6],[10]. By introducing $2(n+1)$ non-negative auxiliary σ_i and ω_i $(i=0,1,2\ldots n)$, the minimization of (30) is equivalent to the minimization of

$$\bar{J}' = L \sum_{i=0}^{n} \xi_i (\sigma_i + \omega_i) \qquad (31)$$

with the constraints

$$\left. \begin{array}{l} \sum_{j=0}^{m} c_{ij} u_j - q_i^* = \sigma_i - \omega_i \\ \sigma_i \geq 0, \quad \omega_i \geq 0 \end{array} \right\} (i = 0,1,\ldots,n) \qquad (32)$$

and $U_1 \leq u_j \leq U_2 \quad (j = 0 \div m)$

Thus, with any u_j , the minimum value of (31) is attained by setting $\omega_i = 0$ if

$\sum_{j=0}^{m} c_{ij} u_j - q_i^*$ is non-negative and $\sigma_i = 0$ if

$\sum_{j=0}^{m} c_{ij} u_j - q_i^*$ is negative. Then, clearly

$$\min \bar{J} = \min \bar{J}'$$

Thus, we can replace the solution of (27) with the constraint (28) by minimizing the problem (31) with the constraint (32).

By using the simplex method in [2],[4],[6] the optimal solution of (31), (32) can be obtained by using numerical method after a finite number of iterations.

C. Calculate the temperature of the furnace v(t) and the temperature distribution in the object q(x,t)

a). Calculate the temperature of the furnace v(t)

We know that $v(t)$ and $u(t)$ have the relation:

$$T.\dot{v}(t) + v(t) = k.u(t-\tau) \qquad (33)$$

or

$$\dot{v}(t) = \frac{k.u(t-\tau) - v(t)}{T} = f(\mathrm{v}, u) \qquad (34)$$

Based on improved Euler formula, we have:

$$v(j+1) = v(j) + l.f(u, v(j))$$

$$v(j+1) = v(j) + \frac{t_f}{m} . \left[\frac{k.u(j) - v(j)}{T} \right] \qquad (35)$$

where $l = t_f / m$; t_f is the allowed burning time (s); with m is the number of time intervals, T is the time constant of the furnace.

After transforming, we get

$$\mathrm{v}(j+1) = \frac{k.l.u(j) + v(j)[T-l]}{T} \qquad (36)$$

or

$$v(j) = \frac{k.l.u(j-1) + v(j-1).[T-l]}{T} \qquad (37)$$

with $j = 0 \div m$

So, when knowing $u^*(t)$ we can calculate $v(t)$ from Eq. (37).

b). Calculate the temperature distribution in the object q(x,t)

To calculate $q(x,t)$ when knowing $u^*(t)$, we use the privious calculated results. Here also use the numerical method [2], [3], [4], [6],[9].

From Eq. (25), we have

$$q(x_i, \mathrm{t}_j) = \int_0^t g(x_i, t_j - \tau).u(\tau) d\tau \qquad (38)$$

$i = 0 \div n$; $t = 0 \div t_f$; n is the number of layers of space. According to trapezoidal formula [3], [4]. After calculating, we obtained

$$q(x_i, t) \approx jl \sum_{\varepsilon=0}^{j\delta} \xi_\varepsilon . g(x_i, jl - \tau_\varepsilon).u(\tau_\varepsilon) \qquad (39)$$

IV. SOME SIMULATION RESULTS

After building the algorithms and establishing the control programs, we have proceeded to run the simulation programs to test calculating programs.

A. The simulation for a flat-slab of Carbon steel

- The physical parameters of the object

- *The heat transfer coefficient* $\alpha = 335$ *(w/m^2.^{0}C)*
- *The heat conducting coefficient* $\lambda = 56$ *(w/m.^{0}C)*
- *The temperature conducting factor* $a=1.03*e^{-5}$ *(m^2/s)*
- *The thickness of the object* $L= 0.2$ *(m)*

- The parameters of the furnace

- *The time constant* $T = 1200$ *(s)*
- *The time delay of the furnace* $\tau = 130$ *(s)*
- *The static transfer coefficient of the furnace* $k = 5$

- The desired temperature distribution $q^* = 300^0C$
- The period of heating time $t_f = 7200$ (s)
- Limit the temperature of furnace $u(t) \leq 750^0$C
- Limit the temperature of flat-slab surface:

$$q(0,t) \leq 600\ ^0\text{C}$$

- Limit under voltage: $U_1=125$ (V)
- Limit upper voltage: $U_2=205$ (V)

With these parameters, the coefficient *Bi* is calculated as follows:

$$Bi = \alpha.\text{L}/\lambda = 335.0,2/56 \approx 1,2$$

Thus, the flat-slab of Carbon steel is a thick object because the coefficient *Bi* is greater than 0.5.

To calculate the optimal heating process, we choose $n = 4$ and $m=16$. After the simulation, we have result like in Figure 1.

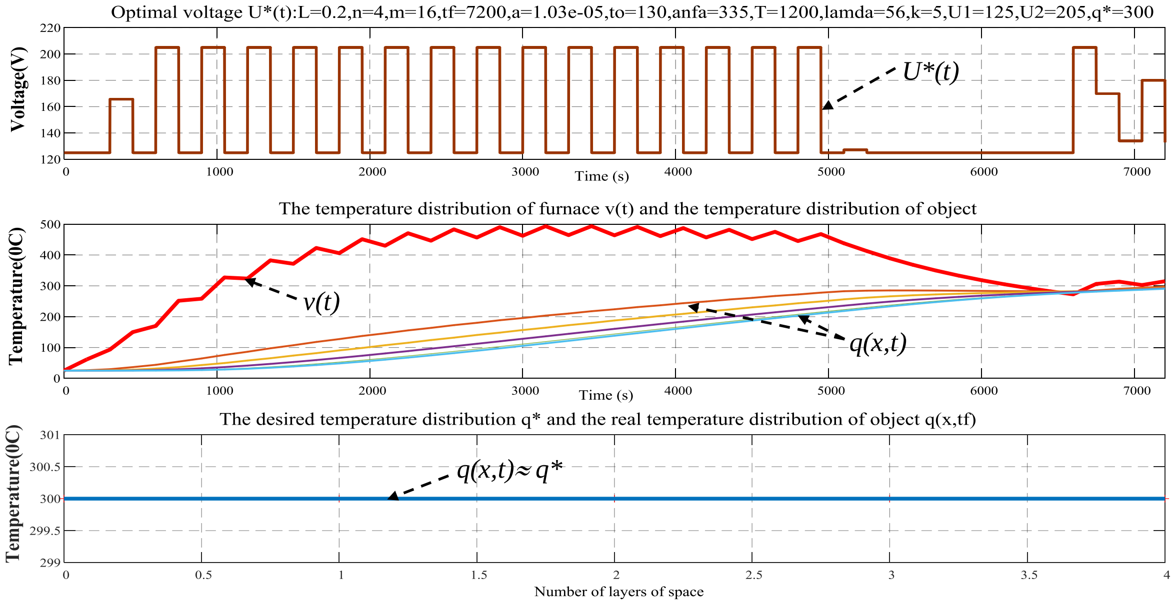

Fig 1. The optimal heating process for a flat-slab of Carbon steel with $q^*=300^0$C

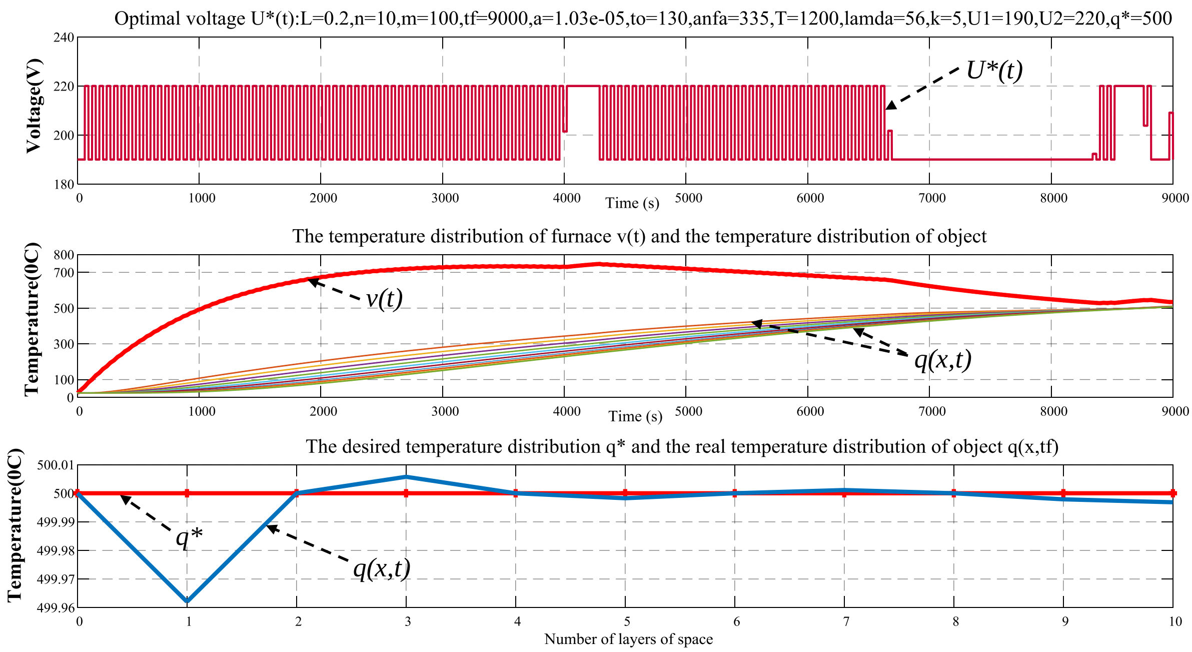

Fig 2. The optimal heating process for a flat-slab of Carbon steel with q^*=500^{0}C

Remarks

In Fig. 1, we can see that at the time $t=t_f=7200(s)$, the temperature distributions of the layers in a flat-slab of Carbon steel $q(x,t_f)$ is approximately equal $q^*=300^0C$ with the error of objective function J as $e \approx 0$. Therefore, the optimal solution has been testified.

In Fig. 2, we increase the desired temperature distribution q^* from 300^0C to 500^0C, so we have to increase the period of heating time t_f from $t_f=7200(s)$ to $t_f=9000(s)$ as well as increase limit under and upper voltage U_1, $U2$.

To calculate the optimal heating process, we choose $n = 10$ and $m=100$

We can also see that at the time $t=t_f=9000(s)$, the temperature distributions of the layers in a flat-slab of Carbon steel $q(x,t_f)$ is approximately equal $q^* \approx 500^0C$ with the error of objective function J as $e \approx 0.05$ Therefore, the optimal solution has been testified.

B. The simulation for a flat-slab of Samot

- The physical parameters of the object

- *The heat transfer coefficient* $\alpha = 60$ $(w/m^2.{}^0C)$
- *The heat conducting coefficient* $\lambda = 0.955$ $(w/m.{}^0C)$
- *The temperature conducting factor* $a=4.86*e^{-7}(m^2/s)$
- *The thickness of the object* $L= 0.03$ *(m)*

- The parameters of the furnace

- *The time constant* $T = 1200$ *(s)*
- *The time delay of the furnace* $\tau = 130$ *(s)*
- *The static transfer coefficient of the furnace* $k = 5$

- The desired temperature distribution $q^* = 300\ {}^0C$
- The period of heating time $t_f = 4200$ (s)
- Limit the temperature of furnace $u(t) \leq 700^0$C
- Limit the temperature of flat-slab surface
  $q(0,t) \leq 500\ {}^0$C
- Limit under voltage: $U_1=125$ (V)
- Limit upper voltage: $U_2=205$ (V)

With these parameters, the coefficient Bi is calculated as follows:

$$Bi = \alpha.L/\lambda = 60.0,03/0.955 \approx 1,9$$

Hence, the flat-slab of Samot is also a thick object because the coefficient Bi is greater than 0.5.
To calculate the optimal heating process, we choose $n = 4$ and $m=16$. After the simulation, we have results like in Figure 3 and Figure 4.

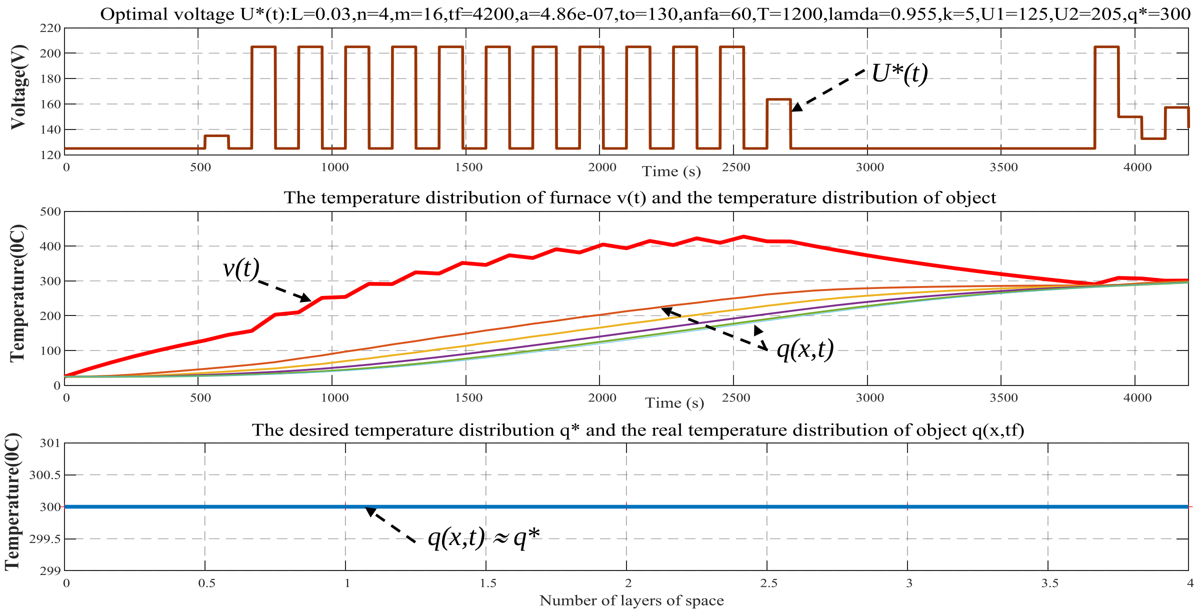

Fig 3. The optimal heating process for a flat-slab of Samot with $q^*=300^0$C

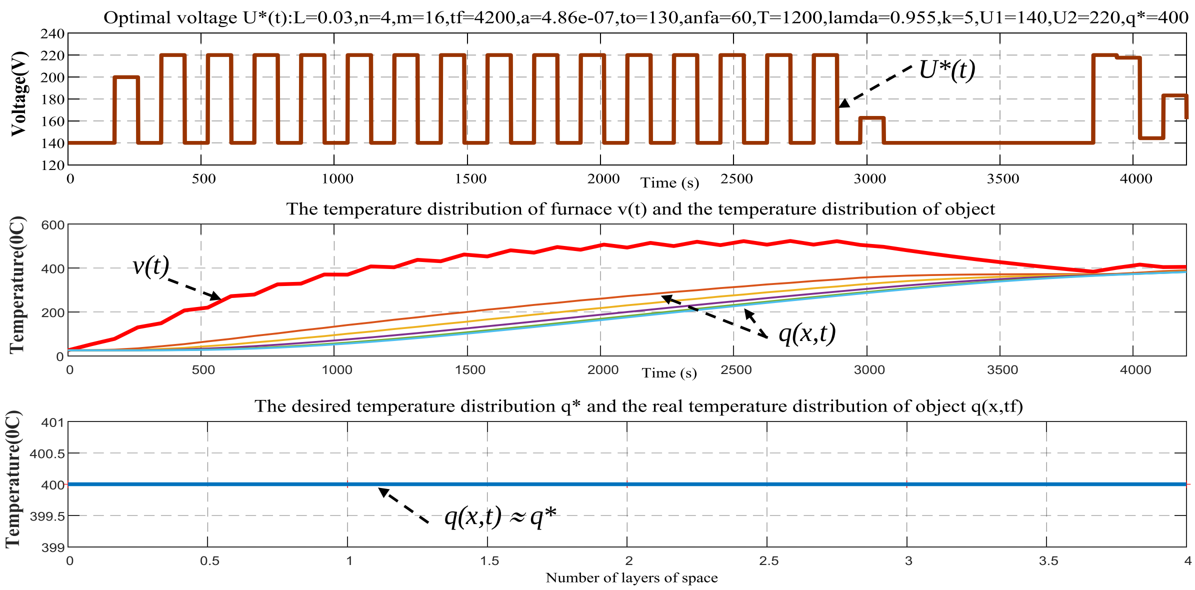

Fig 4. The optimal heating process for a flat-slab of Samot with $q^*=400^0$C

Remarks

In Fig. 3, we can see that at the time $t=t_f=4200(s)$, the temperature distributions of the layers in a flat-slab of Samot $q(x,t_f)$ is also approximately equal $q^*=300^0C$ with the error of objective function J as $e \approx 0$. So, the optimal solution has been testified.

In Fig. 4, we also increase the desired temperature distribution q^* from 300^0C to 400^0C. The period of heating time t_f is not change, we only increase the limit under and upper voltage U_1, $U2$. After simulating, we see that at the time $t=t_f=4200(s)$, the temperature distributions of the layers in a flat-slab of Samot $q(x,t_f)$ is also approximately equal $q^*=400^0C$ with the error of objective function J as $e \approx 0$. Therefore, the optimal solution has been testified, too.

In Figures from Fig.1 to Fig.4.

where $U^*(t)$ is the optimal control signal (optimal voltage) of the furnace; $v(t)$ is the temperature of the furnace; $q(x,t)$ is temperature distribution of the flat-slab, including the temperature of the two surfaces and the temperature of the inner layers of the flat-slab. q^* is the desired temperature distribution

VI. CONCLUSIONS

The paper has solved a system consitting of parabolic-type partial differential equation with boundary condition type-3 combined with a time-delayed ordinary differential equation. An optimal solution for a distributed parameter system with time-delay has been defined by using a numerical method. Algorithms and optimal calculating program have been accuracy. Then, we have proceeded to run the simulations on a flat-slab of Carbon steel and a flat-slab of Samot in order to test the algorithms once again.

The analytical results obtained here are available not only for the heat conduction system but also for an arbitrary distributed parameter system, so long as the state of a system is expressed as (17) or (18) .

The methods described here appear to be of wider application. For example, the technique of replacing the minimizzation of a functional by the minimization of a functions of many variables is also applicable for solving the optimal control problem in lumped-parameter system.

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