Comparison of two Replacing Methods a delayed Object in Optimal Control Problem for a Distributed Parameter System

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Abstract: *The delayed control objects often meet in many different fields such as industry, transport, transportation, military... Normally, when designing the controller, if the object is the delayed first order inertia system which is approximated by two systems of the first order inertia, this often leads to the large error if the delayed time (τ) is significantly large compared to its time constant (T). This paper presents a research comparing the accuracy of the solution when replacing a delayed object by Taylor approximation model and first-order Pade approximation model (Pade-1) so as to solve the optimal control problem for a distributed parameter system with delayed time. The system is applied to a specific one-sided heat-transfer system in a heating furnace to control temperature for a flat-slab following the most accurate burning standards.*

Keywords: - *optimal control; distributed parameter systems; delay; numerical method; Taylor approximation; Pade approximation*

I. INTRODUCTION

Theoretically, Taylor approximation and Pade approximation [1] have been studied for a long time and its mainly applycation is to find the solution of differential algebraic equations. Pade approximation can be offered the functional approximation having more advantages than Taylor expansion, especially with objects have large delayed time (τ) compare to its time constant (*T*) [4].

The paper gives two replacing methods for a delayed object by using Taylor approximation model and Pade-1 approximation model so as to solve the optimal control problem for a distributed parameter system with delayed time, typically for delayed objects with distributed parameter is heat transfer process.

Algorithms and simulation results have shown that depending on the relationship between (τ) and (*T*), which approximation form is best used.

II. THE PROBLEM OF OPTIMAL CONTROL

1. The object model

As a typical distributed parameter system, the one-sided heat conduction system is considered. The process of one-sided heating of the objects which have flat-slab shape in a furnace is described by the parabolic-type partial differential equation, as follows in [2], [3, [4], [5].

$$a\frac{\partial^2 q(x,t)}{\partial x^2}=\frac{\partial q(x,t)}{\partial t} \tag{1}$$

where *q(x,t)*, the temperature distribution in the object, is the output needing to be controlled, depending on the spatial coordinate x with $0 \le x \le L$ and the time *t* with $0 \le t \le t_f$, a is the temperature-conducting factor (m^2/s), *L* is the thickness of object (m), t_f is the allowed burning time (s)

The initial and boundary conditions are given in [2],[3],[4],[5].

$$q(x,0)=q_0(x)=const \tag{2}$$

$$\lambda\left.\frac{\partial q(x,t)}{\partial x}\right|_{x=0}=\alpha\left[q(0,t)-v(t)\right] \tag{3}$$

$$\left.\frac{\partial q(x,t)}{\partial x}\right|_{x=L}=0 \tag{4}$$

with α as the heat-transfer coefficient between the furnace space and the object (W/m^2.^{0}C), λ as the heat-conducting coefficient of material (W/m.^{0}C), and *v(t)* as the temperature of the furnace respectively (^{0}C).

The relationship between the provided voltage for the furnace *u(t)* and the temperature of the furnace *v(t)* is usually the first order inertia system with delayed-time as in [2], [3, [4], [5].

$$T.\frac{dv(t)}{dt}+v(t)=k\,.u(t-\tau) \tag{5}$$

where T is the time constant; τ is the delayed time; k is the static transfer coefficient; *v(t)* is the temperature of the furnace and *u(t)* is the provided voltage for the furnace (controlled function of the system).

2. The objective function and the constrained conditions

In this case, the problem is set out as follows: we have to determine a control function *u(t)* with ($0 \le t \le t_f$) so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It means at the end of the heating process to ensure temperature uniformity throughout the whole material:

$$J\left[u(t)\right]=\int_0^L\left[q^*(x)-q(x,t_f)\right]^2 dx\rightarrow \min \quad (6)$$

The constrained conditions of the control function is:

$$U_1\le u(t)\le U_2 \quad (7)$$

with U_1,U_2 are the under and upper limit of the supplied voltage respectively (V).

III. THE SOLUTION OF PROBLEM

The process of finding the optimal solution includes 2 steps:

- *Step 1*: Find the relationship between *q(x,t)* and the control signal *u(t)*. Namely, we have to solve the equation of heat transfer (relationship between *v(t)* and *q(x,t)*) with boundary condition type-3 combined with ordinary differential equation with delayed time (relationship between *u(t)* and *v(t)*)

- *Step 2*: Find the optimal control signal $u^*(t)$ by substituting *q(x,t)* found in the first step into the function (6), after that finding optimal solution $u^*(t)$.

1. Find the relationship between q(x,t) and the control signal u(t)

To solve the partial differential equation (1) with the initial and the boundary conditions (2), (3), (4), we apply the Laplace transformation method with the time parameter *t*. On applying the transform with respect to *t*, the partial differential equation is reduced to an ordinary differential equation of variable *x*. The general solution of the ordinary differential equation is fitted to the boundary conditions, and the final solution is obtained by the application of the inverse transformation.

Transforming Laplace (1), we obtained:

$$a\frac{\partial^2 Q(x,s)}{\partial x^2}=sQ(x,s) \quad (8)$$

where: $Q(x,s)=\mathbf{L}\{q(x,t)\}$

After transforming the boundary conditions (3), (4), we have:

$$\lambda\frac{\partial Q(x,s)}{\partial x}\bigg|_{x=0}=\alpha\left[Q(0,s)-V(s)\right] \quad (9)$$

$$\frac{\partial Q(x,s)}{\partial x}\bigg|_{x=L}=0 \quad (10)$$

To solve this problem, [2] replaced delayed object in Eq. (5) satisfy the condition $T/\tau \ge 10$ by the first order inertia system following Taylor approximation, [4], [5] replaced delayed object in Eq. (5) satisfy the condition $6 \le T/\tau < 10$ by Pade-1 approximation Transforming Laplace Eq. (5), we obtained:

- Following Taylor, Eq. (5) becomes:

$$(Ts+1)V(s)=k.U(s).e^{-\tau s}\approx k.\frac{U(s)}{1+\tau s} \quad (11)$$

- Following Pade-1, Eq. (5) becomes:

$$(Ts+1)V(s)=k.U(s).e^{-\tau s}\approx k.U(s)\frac{2-\tau s}{2+\tau s} \quad (12)$$

where: $V(s)=\mathbf{L}\{v(t)\}$; $U(s)=\mathbf{L}\{u(t)\}$ (13)

The general solution of (8) is:

$$Q(x,s)=A(s)sh\left(\sqrt{\frac{s}{a}}.x\right)+B(s)ch\left(\sqrt{\frac{s}{a}}.x\right) \quad (14)$$

where: *A(s); B(s)* are the parameters need to be find. From boundary conditions (3), (4), we calculated:

$$A(s)=\frac{-\alpha V(s)sh\sqrt{\frac{s}{a}}.L}{\lambda\sqrt{\frac{s}{a}}.sh\sqrt{\frac{s}{a}}.L+\alpha.ch\sqrt{\frac{s}{a}}.L} \quad (15)$$

$$B(s)=\frac{\alpha V(s)ch\sqrt{\frac{s}{a}}.L}{\lambda\sqrt{\frac{s}{a}}.sh\sqrt{\frac{s}{a}}.L+\alpha.ch\sqrt{\frac{s}{a}}.L} \quad (16)$$

Substituting (15) and (16) into (14), and from (11), (12), after transforming, we have:

- Function *Q(x,s)* (following Taylor)

$$Q(x,s)=U(s)\frac{k.ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1)(\tau s+1)\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}sh\sqrt{\frac{s}{a}}.L+ch\sqrt{\frac{s}{a}}.L\right)} \quad (17)$$

Putting

$$G(x,s)=\frac{k.ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1)(\tau s+1)\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}sh\sqrt{\frac{s}{a}}.L+ch\sqrt{\frac{s}{a}}.L\right)} \quad (18)$$

We have: *Q(x,s) = G(x,s) .U(s)* (19)

- Function *Q(x,s)* (following Pade-1)

$$Q(x,s)=\frac{U(s).k.\left(1-\frac{\tau s}{2}\right).ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1).\left(1+\frac{\tau s}{2}\right).\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}.sh\sqrt{\frac{s}{a}}.L+ch\sqrt{\frac{s}{a}}.L\right)} \quad (20)$$

Putting

$$G(x,s)=\frac{k.\left(1-\frac{\tau s}{2}\right).ch(L-x)\sqrt{\frac{s}{a}}}{(Ts+1).\left(1+\frac{\tau s}{2}\right).\left(\lambda\frac{\sqrt{\frac{s}{a}}}{\alpha}.sh\sqrt{\frac{s}{a}}.L+ch\sqrt{\frac{s}{a}}.L\right)} \quad (21)$$

We also have: *Q(x,s) = G(x,s) .U(s)* (22)

From (19) and (22), according to the convolution theorem, the inverse transformation of (19) and (22) is given by

$$q(x,t) = g(x,t) * u(t) \quad (23)$$

We can write

$$q(x,t) = \int_0^t g(x,\tau).u(t-\tau)d\tau \quad (24)$$

or

$$q(x,t) = \int_0^t g(x,t-\tau).u(\tau)d\tau \quad (25)$$

where

$$g(x,t) = \mathbf{L}^{-1}\{G(x,s)\} \quad (26)$$

Therefore, if we know the function *g(x,t)*, we will be able to calculate the temperature distribution *q(x,t)* from control function *u(t)*. To find *q(x,t)* in (25), we need to find the function (26). Using the inverse Laplace transformation of function *G(x,s)* we have the following result:

- Function g*(x,t)* (following Taylor)

$$g(x,t) = \frac{k.k_0^2.\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(1-\tau k_0^2\right)\left[\cos\left(\frac{k_0 L}{\sqrt{a}}\right) - \frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0 L}{\sqrt{a}}\right)\right]}.e^{-k_0^2 t} + \frac{k.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1 L}{\sqrt{a}}\right) - \frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1 L}{\sqrt{a}}\right)\right]}.e^{-k_1^2 t} + \sum_{i=2}^{\infty}\frac{2\alpha.k.\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(1-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right) + \frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2 t} \quad (27)$$

with $k_0 = 1/\sqrt{T};\ k_1 = 1/\sqrt{\tau};$

- Function g*(x,t)* (following Pade-1)

$$g(x,t) = \frac{k.k_0^2\left(2+\tau k_0^2\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(2-\tau k_0^2\right)\left[\cos\left(\frac{k_0 L}{\sqrt{a}}\right) - \frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0 L}{\sqrt{a}}\right)\right]}.e^{-k_0^2 t} + \frac{2k.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1 L}{\sqrt{a}}\right) - \frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1 L}{\sqrt{a}}\right)\right]}.e^{-k_1^2 t} + \sum_{i=2}^{\infty}\frac{2\alpha.k\left(2+\tau.\Psi_i^2\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(2-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right) + \frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2 t} \quad (28)$$

with $k_0 = 1/\sqrt{T};\ k_1 = \sqrt{2/\tau}$

In Eq. (27) and Eq. (28):

- α is the heat-transfer factor ($W/m^2.^0C$)
- λ is the heat-conducting factor of object ($W/m.^0C$)
- L is the thickness of object (m),
- a is the temperature-conducting factor (m^2/s)
- τ is the delayed time of the furnace (s)
- T is the time constant of the furnace (s)
- k is the static transfer coefficients of the furnace
- Ψ_i is calculated from the formula:

$$\Psi_i = \phi_i\sqrt{a}\,/\,L \quad (29)$$

- ϕ_i is the solution of the equation:

$$\phi_i.tg\phi_i = \alpha L/\lambda = B_i$$

(30)

- B_i is the coefficient BIO of the material.

Conclusions:

We have solved a system of parabolic-type partial differential equation with boundary conditions of type-3 (the relationship between *v(t)* and *q(x,t)*) combined with the ordinary differential equation with time delay (the relationship between *u(t)* and *v(t)*).

Thus, if we are not interested in the optimal problem, we can calculate the temperature field in the object when knowing the supplied voltage for the furnace (The problem knows the shell to find the cores), as follows:

The relationship between the supplied voltage for the furnace u(t) and the temperature field distribution in the object q(x,t):

$$q(x,t) = g(x,t) * u(t) = \int_0^t g(x,t-\tau).u(\tau)d\tau \quad (31)$$

with t_f is the allowed burning time (s).

2. Find the optimal control signal $u^*(t)$ by using numerical method

To find the $u^*(t)$, we have to minimize the objective function (6), it means:

$$J[u(t)] = \int_0^L \left[q^*(x) - q(x,t_f)\right]^2 dx \rightarrow \min \quad (32)$$

or

$$J[u(t)] = \int_0^L \left[q^*(x) - \int_o^{t_f} g(x,t_f-\tau)u(\tau)d\tau\right]^2 dx \rightarrow \min \quad (33)$$

with $q^*(x)$ is the desired temperature distribution; $q(x,t_f)$ is the real temperature distribution of the object at time $t = t_f$.

As calculated in [2], [3], [4], [5] the integral numerial method is used by applying Simson formula to the right-hand side of the objective function (33). The *L*, the thickness of the object, is divided into *n* equal lengths (*n* is an even number).

Similarly, it is applied to the right-hand side of the equation (33). The period of time t_f is devided into m equal intervals that m is an even number, too.
Thus, the optimal control problem is here to find u^*_j in order to minimize the objective function:

$$J[u^*] = L\sum_{i=0}^{n}\xi_i\left[q_i^* - \sum_{j=0}^{m}c_{ij}.u_j\right]^2 \quad (34)$$

The constrained conditions of the control function:

$$U_1 \le u_j \le U_2 \ \ (j = 0,1,2 \ldots m) \quad (35)$$

The performance index (34) is a quadratic function of the variables u_j with constraints (35) are linear, the problem becomes a quadratic programming problem. This problem can be obtained by using numerical method after a finite number of iterations of computation.

Although a solution of the quadratic programming problem is obtained after a finite number of iterations of computation, but its algorithm is more complicated than that of the simplex method for linear programming. If the performance index is taken as

$$J = \int_0^L \left|q^*(x) - q(x,t_f)\right| dx \quad (36)$$

instead of (32), the linear programming technique can be used directly. On applying the same procedure as mentioned above, the approximate performance index corresponding to (34) is written as

$$J \cong \bar{J} = L\sum_{i=0}^{n}\xi_i\left|q_i^* - \sum_{j=0}^{m}c_{ij}.u_j\right| \quad (37)$$

Hence, we can replace the solution of (34) with the constraint (35) by minimizing the problem (37) with the constraint (35).

By using the simplex method in [2], [3], [4], [5] the optimal solution of (37) can be obtained by using numerical method after a finite number of iterations.

IV. SOME SIMULATION RESULTS

After building the algorithms and establishing the control programs, we have proceeded to run the simulation programs on a Diatomite sample in two cases Pade-1 approximation and Taylor approximation in order to test calculating programs.

4.1. Case 1: when delayed objects satisfy the condition: $T/\tau \ge 10$ in [2], [4]

- The physical parameters of the object

  $\alpha = 60$ (w/m^2.^{0}C); $\lambda = 0.2$ (w/m.^{0}C)

  $a = 3.6*e^{-7}$(m^2/s); $L = 0.04$ (m)
- The parameters of the furnace

  $T = 1200$ (s); $\tau = 80$ (s;) $k = 0.3$
- The desired temperature distribution $q^* = 400^0C$
- The period of heating time $t_f = 5400$ (s)
- Limit the temperature of furnace $u(t) \le 600^0$C
- Limit the temperature of flat-slab surface:

  $q(0,t) \le 500\ ^0$C
- Limit under voltage: $U_1 = 125$ (V)
- Limit upper voltage: $U_2 = 205$ (V)

With these parameters, the coefficient B_i is calculated as follows: $B_i = \alpha.L/\lambda = 60.0,04/0.2 = 12$

Thus, the flat-slab of Diatomite is a very thick object because the coefficient B_i is greater than 0.5.

We have: $T/\tau = 1200/80 = 15 > 10$

To calculate the optimal heating process, we choose $n = 6$ and $m=36$. After the simulation, we have results like in Figure 1 and Figure 2.

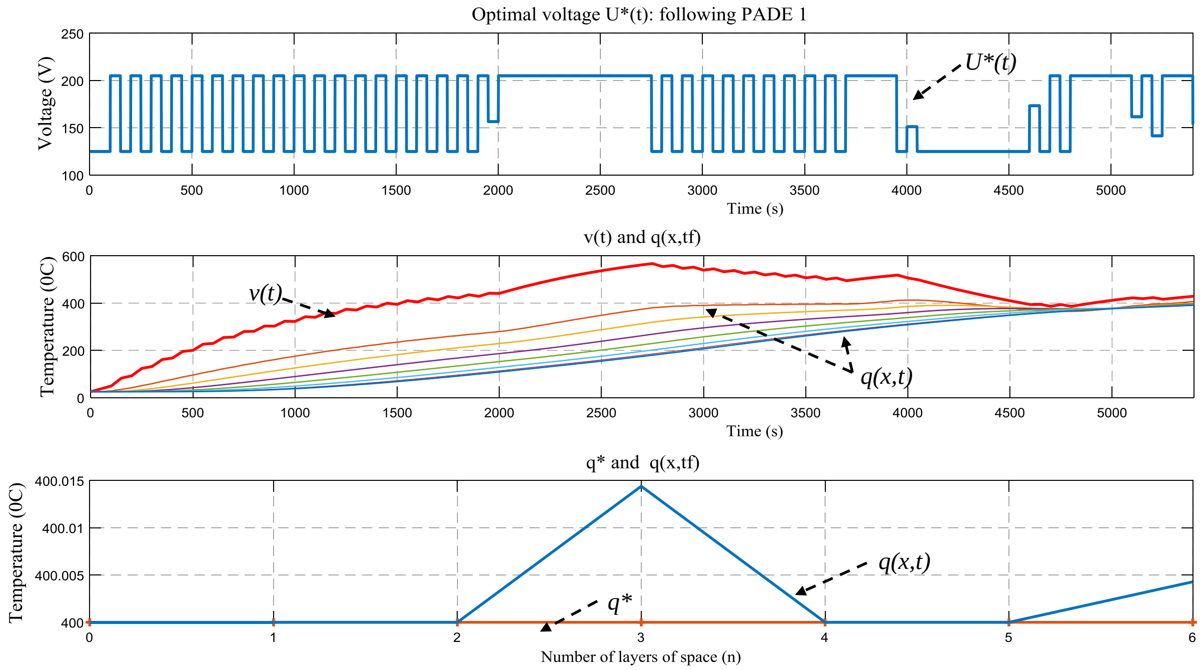

Fig 1. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C (e = 9.6096e-07)

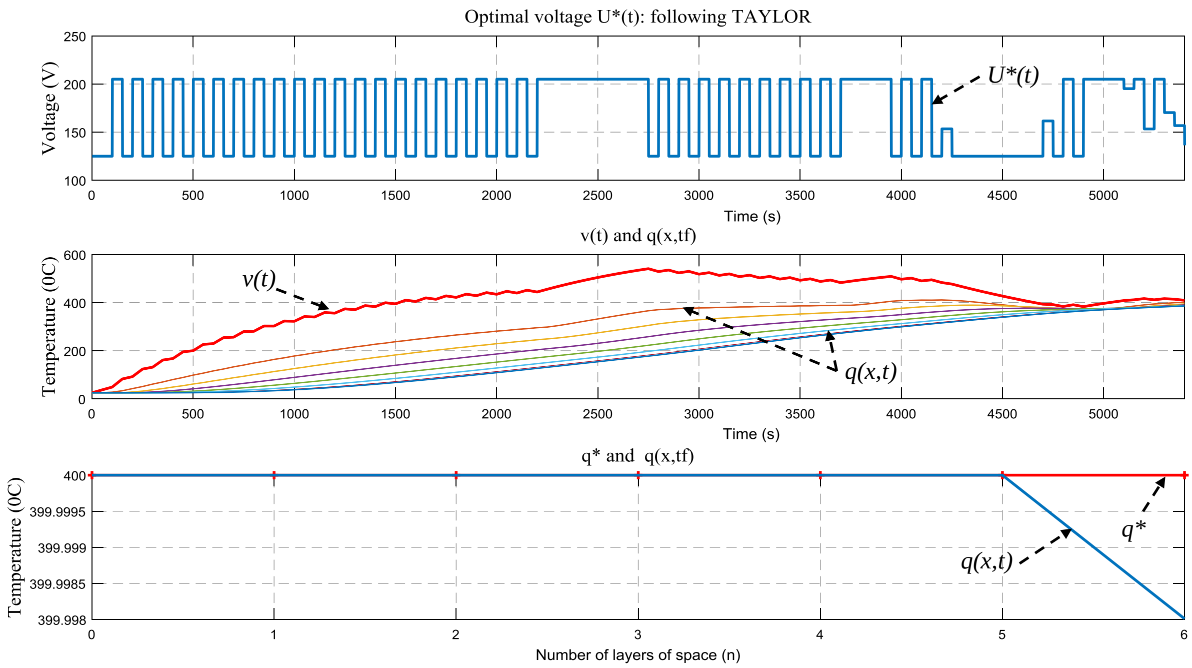

Fig 2. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C (e = 8.7929e-09)

4.2. Case 2: when delayed objects satisfy the condition: $6 \le T/\tau < 10$ in [2], [4]

In the simulation process, we also keep all the parameters as in the case 1 but only change the delayed time τ, in this case for τ = 150 (s), so we have: $T/\tau = 6 \le 1200/150 = 8 < 10$. After the simulation, we also have results like in Figure 3 and Figure 4.

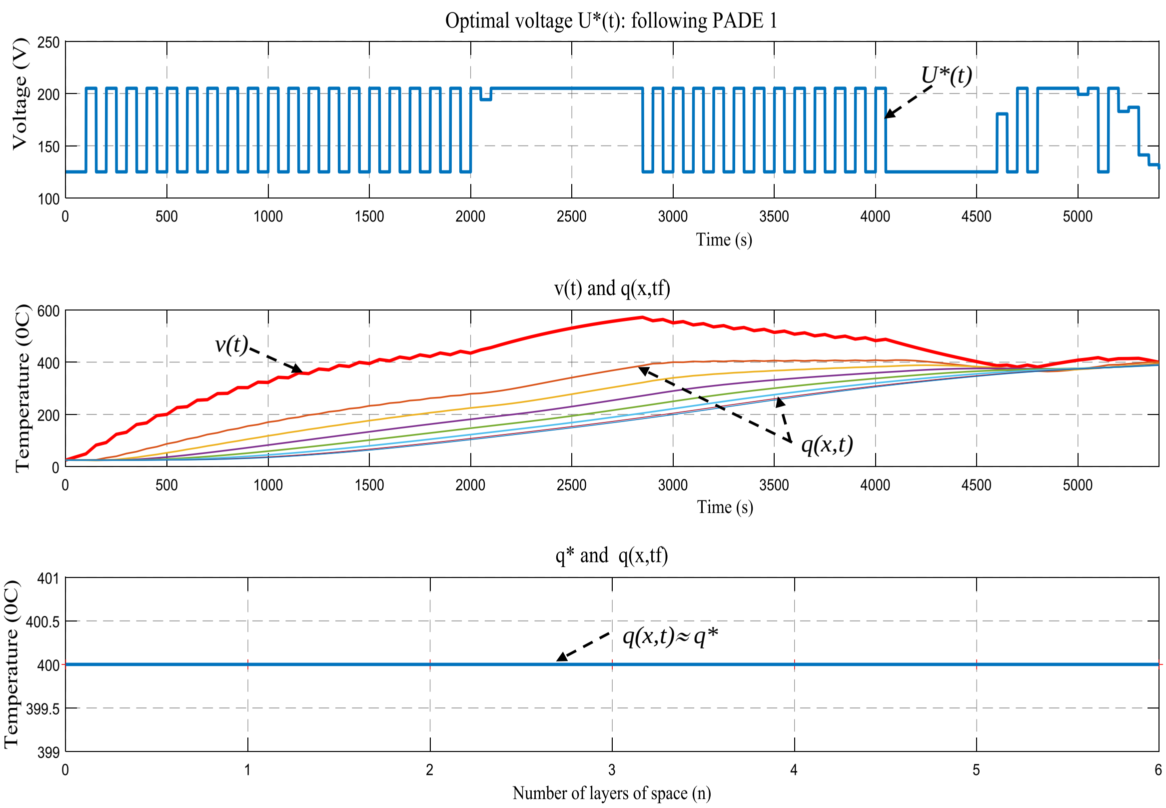

Fig 3. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C ($e \approx 0$)

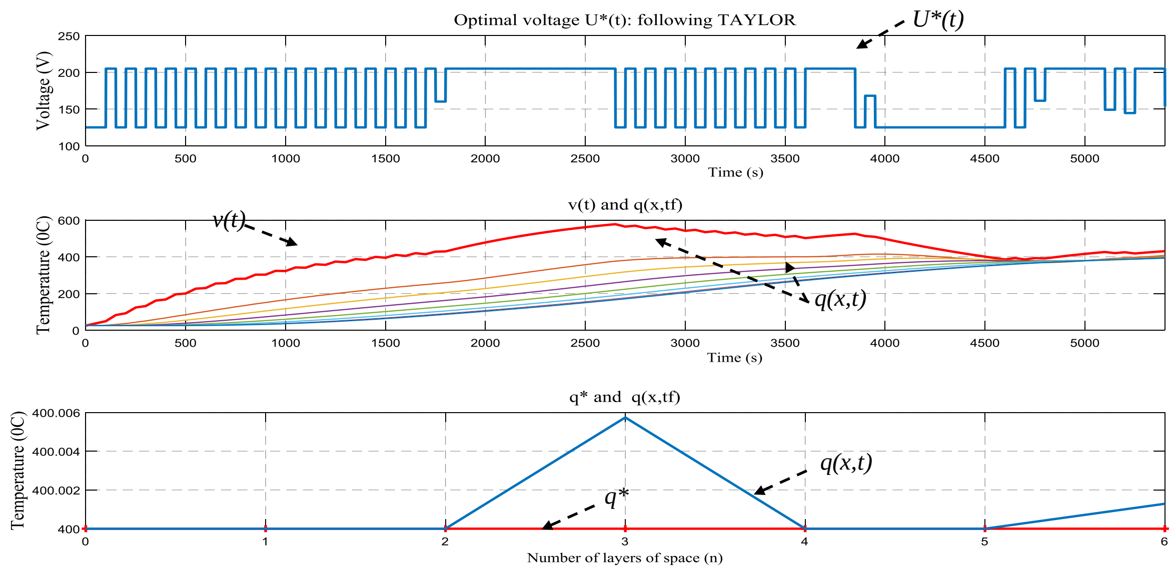

Fig 4. The optimal heating process for a flat-slab of Diatomite with $q^* = 400^0$C (e = 1.5013e-07)

V. COMPARISON OF TWO METHODS

Figure 1 and Figure 2 show that both methods at the time $t=t_f$ =5400s the temperature distribution at layers $q(x,t_f)$ is approximately 400^0C, when approximating according to Taylor, the error of objective function J as e = *8.7929e-09* and according to Pade-1 the error of objective function J as e = *9.6096e-07*. So when the delayed object satisfies the condition $T/\tau \geq 10$, the Taylor approximation will have smaller deviation than the Pade-1 approximation.

Figure 3 and Figure 4 also show that at the time $t=t_f$=5400s the temperature distribution at layers $q(x,t_f)$ is also approximately 400^0C, but when approximating according to Taylor, the error of objective function J as e = *1.5013e-07* and according to Pade-1 the error of objective function J as $e \approx 0$. Thus, when the delayed object satisfies the condition $6 \leq T/\tau < 10$, the Pade-1 approximation will have smaller deviation than the Taylor approximation.

VI. CONCLUSIONS

The paper presented two replacing methods for a delayed object by using Taylor approximation and Pade-1 approximation in order to solve the optimal control problem for a distributed parameter system with delayed time. The system is applied to a specific one-sided heat-transfer system in a heating furnace to control temperature for a flat-slab following the most accurate burning standards**.** We have found an optimal voltage $u^*(t)$ so as to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It also means that at the end of the heating process to ensure temperature uniformity throughout the whole material.

The simulation results have shown the correctness of the algorithms and the two methods have also shown that depending on the relationship between (τ) and (T), which approximation form is best used.

Namely, when delayed object has (τ) and (T) satisfying $T/\tau \geq 10$, using Taylor approximation will have higher accuracy. If delayed object satisfies the condition $6 \leq T/\tau < 10$, using Pade-1 approximation will have higher accuracy.

From the above conclusions, the problems of control object identification and controller design will be corrected accordingly.

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