SSRG International Journal of Economics and Management Studies
ISSN: 2393 – 9125 / http://doi.org/10.14445/23939125/IJEMS-V7I12P106
Volume 7 Issue 12, 43-46, December 2020

Original Article

A Conditional Probability of Default Under the Influence of Both Systematic and Idiosyncratic Components

Camilo Sarmiento
Inter-American Development Bank, Washington, DC

Received Date: 03 November 2020
Revised Date: 15 December 2020
Accepted Date: 17 December 2020

Abstract - *In this article, we relax the assumption of a fully granular portfolio underpinning the Basel II risk weights for credit risk in determining capital requirements. To do so, we model losses under an adverse scenario that stems from both the systematic and idiosyncratic components. The generalization requires the use of a numerically derived distribution function via simulation. Unlike previous work, our method is sufficiently flexible to accommodate a fat-tailed distribution in the idiosyncratic component. The idiosyncratic component is pertinent for a non-granular portfolio.*

Keywords - *Probability of Default, Adverse Scenario, Systematic, Idiosyncratic.*

I. INTRODUCTION

The main determinant of capital requirements is credit risk, which generally arises from a counterparty failing to meet a repayment commitment on its outstanding debt. Such credit events, however, tend to correlate with adverse macroeconomic (systematic) events. The measurement of credit risk under adverse scenarios is thus the main objective in enterprise risk management. For example, regulators and senior management of major US banks rely on stress testing to guarantee that the bank holding company has sufficient capital to continue operations throughout times of economic and financial stress.

The determination of capital requirements for credit risk hinges on loss forecasts. For example, the Probability of Default (PD) and loss given default (LGD) models provide the framework for the projection of losses under different adverse scenarios that include rare events. Such projections can then be used to determine capital levels that ensure the bank's solvency under different levels of stress.

The use of agency ratings (e.g., Standard & Poor's) in determining capital requirements are common for projecting portfolio classes' losses with little or no default history. In these instances, each counterparty's credit rating can be mapped to an average probability of default per rating. Such data is available for over 30-years of data, and rating agencies report it (e.g., see S&P Global Ratings, 2020). Therefore, if a bank has a portfolio with few default events, it can assign a probability of default to each rated counterparty.

An average probability of default, however, provides little help in the determination of capital requirements. In a seminal contribution, Vacisek (2002) solves the puzzle by mapping the unconditional (average) probability to a conditional probability of default. For example, the conditional probability of default can be defined at a stress event, a 99.9^{th} percentile event for the systematic (macroeconomic) component that drives default.

Vacisek's model's main assumption is that a default occurs if the stochastic component of the asset value of a given entity drops below a certain threshold. Vacisek's model uses information on asset correlation, and the PD's unconditional mean to derive the conditional PD under stress. However, the derivation of the conditional PD assumes a fully granular portfolio. Vacisek's model underpins the Basel framework (Basel II risk weights) for credit exposures.

Yet, the idiosyncratic component is a major factor in explaining default (see Hilscher and Wilson, 2016). Gordy (2004) introduces concentration risk as the additional risk associated with a second-order Taylor expansion around the idiosyncratic component. Therefore, the approach incorporates the additional fraction of risk from the idiosyncratic component to Vacisek's formula. The Basel Committee on Banking Supervision (2014) defines applications in which such an add-on is needed for regulatory capital.

In this paper, we relax the assumption of a fully granular portfolio by departing from the conditional PD's analytical representation and modeling losses that stem from both the systematic and idiosyncratic components. The generalization requires the use of a numerically derived distribution function via simulation. The testing of the derived distribution illustrates the effects of modeling both the systematic and idiosyncratic components. Unlike previous work, our method is sufficiently flexible to accommodate a fat-tailed distribution for the idiosyncratic component.

II. CONDITIONAL PROBABILITY OF DEFAULT UNDER STRESS

Consider a portfolio of N sovereign assets with expected losses at time t that are defined as follows:

(1) $Loss(S_t) = \sum_{i=1}^{N} w_{it} \times PD_{it} \times LGD$

where

w_{it} = weight of asset i in the portfolio;

PD_{it} = probability of default of asset i; and

LGD = Loss given default.

For low default incidence portfolios, the PD is generally inferred from the rating of the counterparty. For example, the PD for a *Ba* rated exposure is the historical (through the cycle) default rate.

Vacisek (2002) maps the unconditional (average) PD to a conditional probability. To do so, Vacisek assumes that the asset value of a given obligator i has a stochastic component,X_{it}, that follows the stochastic process:

(2) $X_{it} = S_t\sqrt{p} + Z_{it}\sqrt{1-p}$

and

$S_t \sim$ N(0,1) and $Z_{it} \sim$ N(0,1)

where the systematic component is S_t; the idiosyncratic component is Z_{it}; and the asset correlation between obligators is p. Generally, S_t relates to an economic index and will depend on the sector. The asset correlation, p, determine the relative importance of the systematic component.[1] Empirical estimates for asset correlation range from .10 to .25 (e.g., see Zhan, Zhu, and Lee, 2008).

[1] The weights of the idiosyncratic versus systematic component depends on the asset class and these weights differs from those weights on explaining default. Reinhart and Rogoff (2005) underscore the role of a systematic factor in repeated sovereign default events.

Consistent with Merton (1977), the default event occurs if the random component of the stochastic component X_{it} falls below a threshold C. Therefore, the unconditional PD is:

(3) $P(X_{it} < C) = [\varphi^{-1}(C)] = PD$

where φ is the distribution function of S_t. As a Corollary, the conditional probability of default for a given realization of the systematic component S_t is:

(4) $CP = P(X_{it} < C|S_t) =$

$$\varphi[(\varphi^{-1}(PD) - S_t\sqrt{p})/\sqrt{1-p}]$$

where CP is the conditional PD.

When setting the adverse event for S_t in (4) to correspond to the 99.9th percentile of the systematic component's underlying distribution, then the conditional PD corresponds to the risk weights under the Basel II formula (see Bank of International Settlements, 2005).

Yet, the analytical representation in (4) assumes a fully granular portfolio and, therefore, the idiosyncratic risk plays no role in determining unexpected losses. However, the idiosyncratic component plays a role if the portfolio is not fully granular. For example, Hilscher and Wilson (2016) estimate that the idiosyncratic component explains up to 80 percent of the firm-level default probabilities.

For a non-granular portfolio, we generalize Vacisek's analytical model by dints of a numerical procedure. That is, the random loss associated with the non-granular portfolio is:

(5) $Loss(S_t) = \sum_{i=1}^{N} w_{it} \times I_{it} \times LGD$

where

$I_{it} = 1, if\ X_{it} < \emptyset^{-1}(PD_i);\ and\ I_{it} = 0,\ otherwise.$

Draws of the stochastic process in (2) under (5) yield a loss distribution, which is numerically derived. The distribution derived from (5) via simulation can then be used to project losses (expected shortfall) under different confidence intervals, analogously to the analytical formula in (4). For example, credit's capital requirements would correspond to the loss associated with a given percentile of the loss distribution in (5), e.g., 99.9th percentile.

Overall, the analytical solution is preferable in the context of a diversified portfolio. For concentrated portfolios, the tail event that corresponds to the 99.9th percentile should be associated with the joint distribution of the systematic and idiosyncratic component. The next section shows the relevance of the distribution derived from (5) via simulation for a non-granular portfolio.

III. PROBABILITY OF DEFAULT UNDER INFLUENCE OF BOTH SYSTEMATIC AND IDIOSYNCRATIC COMPONENTS UNDER STRESS

To examine the impact of modeling both the idiosyncratic and systematic components in (5), we simulate under different assumptions on a hypothetical portfolio with different degrees of granularity. The reported loss for different granularity degrees relates to a tail of the distribution, an adverse event.

The simulation shows the effect of the number of assets (or counterparties) in the portfolio under a baseline asset correlation assumption. In particular, the analysis uses the assumption of an asset correlation of 20%, $\rho = 0.2$ (see Zhan Zhu, and Lee, 2008; the Bank of International Settlements, 2005). For simplicity, all assets are assumed to have a rating of *Baa2,* and, for robustness, we also evaluate results under the assumption of a portfolio rating of *Ba2*.[2]

Tables 1 and 2 reports the conditional loss for an adverse event for a portfolio rated *Baa2* and *Ba2*, respectively. In the tables, all assets are equally distributed, i.e., a portfolio with 40 assets pertains to a portfolio with 40 equally distributed counterparties in the portfolio. Also, for simplicity's sake, we assume an LGD of 100%.

Reported loss estimates in the tables relate to the expected shortfall. Expected shortfall is calculated by averaging all losses in the distribution that are worse than the value at risk of the portfolio at a given level of confidence. For example, the expected shortfall for the 99.9th confidence interval (with an initial rating of *Baa2*) is 5.0% under a fully diversified portfolio (using Vacisek's analytical formula) and 7.3% under a portfolio that comprises only 40 counterparties. Table 2 shows consistent results under an initial rating of *Ba2*.

It is evident from the analysis that the probability of default for a concentrated portfolio is markedly larger under an adverse event. A portfolio comprised of only 15 counterparties has approximately thrice the probability of default of a diversified portfolio at the tail of the distribution.

The analysis also indicates that the effect of concentration is similar for more extreme events, 99.9% confidence interval (CI), relative to less extreme events, 99% CI. Thus, financial planning with a longer time horizon (e.g., 10-years) embeds similar concentration risk in loss projections than a portfolio with a shorter time horizon (e.g., 3-years).

Table 1. Expected Shortfall Loss Associated with Different Degree of Portfolio Concentration for an Initial Rating of *Baa2*

Number of Counterparties	Expected Shortfall (CI 99.9%)	Expected Shortfall (CI 99%)
Fully Granular	5.0%	2.1%
60	5.4%	3.5%
40	7.3%	4.2%
20	11%	6.3%
15	14%	7.7%
10	17%	11%

Table 2. Expected Shortfall Loss Associated with Different Degree of Portfolio Concentration for an Initial Rating of *Ba2*

Number of Counterparties	Expected Shortfall (CI 99.9%)	Expected Shortfall (CI 99%)
Fully Granular	10.9%	6.2%
60	11.8%	6.8%
40	13.6%	8.9%
20	23.1%	13.3%
15	24.5%	15.5%
10	28.6%	17.1%

Lastly, while portfolio concentration is critical, it is not as important as the portfolio's average rating. For example, from Tables 1 and 2, a three-notch downgrade from *Baa2* and *Ba2* for the overall portfolio is equivalent to a mapping from a fully granular portfolio to a concentrated portfolio with 20 counterparties.

Overall, expected tail loss events are significantly larger in more concentrated portfolios. Therefore, for a profit-maximizing institution, a non-granular portfolio is generally too expensive in terms of capital expense, and portfolio diversification is the main task for risk managers. However, for a non-profit lending institution (e.g., a multilateral bank), the presence of a non-granular portfolio is (in some instances) an intrinsic component of the institution's mission. Larger capital requirements are thus generally needed in these institutions.

IV. INCORPORATION OF A MORE FLEXIBLE DISTRIBUTION -- THE T-DISTRIBUTION

The finance literature has explored alternative distributions to the Gaussian distribution (e.g., Sarmiento,

[2] The PD for *Baa2* and *Ba2* ratings are extracted from Moody's historical data set on the average unconditional default rate per rating for the period 1980 to 2019. The average PD for *Baa2* and *Ba2* rated exposures are 0.17% and 0.7%, respectively.

2020). The most commonly used distribution to capture fat tails in finance is the Student t distribution (see Blattberg and Gonedes, 1974). We next explore the use of a t-distribution for the idiosyncratic component of the stochastic process in (2).

In the context of the stochastic process in (2), it is to be expected that the systematic component (e.g., an aggregate macroeconomic indicator) is normally distributed due to the Central Limit Theorem. However, the idiosyncratic process could be expected to feature fatter tails.

Our flexible numerical procedure easily relaxes the assumption of a Gaussian distribution for the idiosyncratic component. To do so, the stochastic process in (2) is generalized as follows:

$$(6) \qquad X_{it} = S_t\sqrt{p} + Z_{it}\sqrt{1-p}$$

and

$$S_t \sim \mathrm{N}(0,1) \text{ and } Z_{it} \sim \mathrm{t(DF)}.$$

Draws of the stochastic process in (6) under (5) yield a loss distribution, which is numerically derived. The distribution derived from (6) via simulation can then be used to project losses (expected shortfall) under different confidence intervals, analogously to the analytical formula in (4). The use of the stochastic process in (6) in lieu of (2) incorporates a more flexible distribution that subsumes fatter tails associated with a t-distribution with low degrees of freedom.

The tail of the numerically derived distribution under the stochastic process in (6) is presented in Table 3 for a t-distribution with 10 degrees of freedom. As in Table 1, Table 3 assumes a *Baa* rated portfolio and an asset correlation of 0.2.

Table 3. Tail Loss Associated with Different Distributions and Degree of Portfolio Concentration for an Initial Rating of *Baa2*

Number of Counterparties	Expected Shortfall (CI 99.9%) Normal Distribution	Expected Shortfall (CI 99.9%) t-distribution with 10 DF
60	5.4%	6.2%
40	7.3%	8.3%
20	11%	13%
15	14%	16%
10	17%	21%

From Table 3, the effects of portfolio concentration are larger if the idiosyncratic component has a fatter tail than the Normal distribution. However, for a t-distribution with 10 DF, the effect of a fatter tail on projected losses (in an adverse scenario) is less punitive than portfolio concentration.

V. CONCLUDING REMARKS

In this article, we relaxed the assumption of a fully granular portfolio that underpins Basel II risk weights for credit risk in determining capital requirements. To do so, we modeled losses under an adverse scenario that stems from both the systematic and idiosyncratic components. The generalization requires the use of a numerically derived distribution function via simulation. Unlike previous work, our method is sufficiently flexible to accommodate a fat-tailed distribution for the idiosyncratic component. The idiosyncratic component is pertinent for a non-granular portfolio, and the issue of portfolio concentration is more severe for assets with fatter tails. Future work should consider using the numerically derived distribution for capital requirements combined with empirically derived measures of degrees of freedom for the idiosyncratic component that varies across asset classes and applications.

VI. REFERENCES

[1] Bank of International Settlements, An Explanatory Note on the Basel II IRB Risk Weight Functions (2005). https://www.bis.org/bcbs/irbriskweight.pdf

[2] Basel Committee on Banking Supervision (2014).

[3] R.C. Blattberg, and N.J. Gonedes, A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices, The Journal of Business 47 (1974) 244–280.

[4] M. Gordy, Granularity Adjustment in Portfolio Credit Risk Measurement, Risk Measures for the 21st Century (Giorgio P. Szegö, Ed.), Wiley. (2004).

[5] J, Hilscher, M. Wilson, Credit Rating, and Credit Risk: Is One Measure Enough?, Managerial Science 63 (2016) 3414 – 3437.

[6] Nishtha Jain, Financial Risk Management: An Introduction SSRG International Journal of Economics and Management Studies 5(1) (2018) 41-43.

[7] Merton, R.C., An Analytical Derivation of the Cost of Deposit Insurance and Loan Guarantees: An Application of Modern Option Theory, Journal of Banking and Finance 1 (1977) 3-12.

[8] C. Reinhart, K. Rogoff, Serial Default, and the "Paradox" of Rich-to-poor Capital Flows, The American Economic Review 94 (2005), 53–58.

[9] C. Sarmiento, Benchmarking Collateral of Triple-A Rated Securities. Applied Economics Letters, 27 (2020) 555-558.

[10] S&P Global Ratings, 2019 Annual Global Corporate Default Study, and Rating Transitions (2020). https://www.maalot.co.il/Publications/TS20200504110435.pdf

[11] O. Vasicek, Loan Portfolio Value, RISK (2002) 160–62.

[12] J. Zhan, F. Zhu, J. Lee, Asset Correlation, Realized Default Correlation, and Portfolio, Credit Risk Modeling Methodology: Modeling Methodology. Moody's K·M·V (2008). https://www.moodysanalytics.com/-/media/whitepaper/before-2011/03-03-08-asset-correlation-realized-default-correlation-and-portfolio-credit-risk.pdf