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Markov Reversibility, Quasi-Symmetry, and Marginal Homogeneity in Cyclothymiacs Geological Successions

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ABSTRACT

Erodogic Markov chains have been popular technique to validate the presence of cyclic succession of facies, facies areas and corresponding environments. A review of published transition tally matrices of sedimentary sequences of different geological time reveals with a few exception the marginal homogeneity and symmetric implying that the underlying depositional processes is a reversible Markov process. Unlike the symmetry model the quasi-symmetry model, a rigorous method implying marginal homogeneity using different parameters for row and column categories and defined as a purely mathematical property of a matrix and can be written as a product of a diagonal and a symmetric matrix. It can then be shown that a Markov sequence is reversible if and only if it has a quasi-symmetric tally matrix. A simple Chi-square (χ2) test for symmetry on the tally matrix is sufficient to determine if an observed matrix is symmetrical and hence whether the Markov cyclicity is reversible or a non-reversible in a sedimentary succession. A new method Kolmogorov criterion is introduced for checking transition matrices of reversible Markov process without requiring knowledge of the stationary probabilities of observed transition probability matrix. The present contribution analyses classical examples with geological data (sedimentary and igneous) of different geological ages around the world to determine if the sequence confirms to a reversible or non-reversible erodogic Markov process and possess quasi-symmetry.

Keywords - *Erodogic Markov Chain, Markov reversibility, Quasi Symmetry, Marginal Homogeneity, Geological Application*

I. INTRODUCTION

It is well established that many geological phenomenon can be measured as data sequences composed of discrete sates (lithology) taken along linear traverses or in drilled log profiles. It was suggested that these data sequences can be described mathematically as Markov chains, following the pioneer work of Vistelius (1949) and co-workers (Vistelius and Faas, 1966) and after a suggestion made by Griffiths (1967) on some application of Markov processes to sedimentation. Starting application was made principally in the field of stratigraphy (Vistelius and Faas, 1966) and these efforts are illustrated by the work of Schwarzacher (1975), Carr (1982), Power and Easterling (1982), and Harper (1984). Concurrently with the efforts in stratigraphy, Vistelius and his coworkers have been applying Markov chains to the texture of granite (Vistelius, 1987), and current applications of the method to sedimentary framework especially coal bearing successions (see Khan et al., 2020, and references therein). However, the approach is useful in that it can often point out crafty relationships in the stratigraphic succession that would not otherwise be noticed or instinctively sought out. More complex Markov processes are possible in which the nature of the Markov dependency includes reference to still earlier beds; or to changes in the dependency relationship with time.

Complex models for contingency table have received an increasing interest in recent years from researchers especially in biological and geological sciences. The general introduction to the statistical models for contingency table is given in Goodman (2002); Agresti (2013) and Ross (2019). Although most books on categorical data analysis in their discussion of two-way cross-classified variables described symmetry, marginal homogeneity and quasi-symmetry. The presence or absence of quasi-symmetry is usually discussed in terms of log-linear models for square contingency tables and there are number of well described examples of tables that have modeled as quasi-symmetric (Bishop et al., 1975; Haberman, 1979, p. 514; Plackett, 1981, p. 81; Sharp and Markham, 2000). Starting from Caussinus (1965) several authors have considered such models from the point of view of both theory and applications (Goodman, 2002; and Kateri, 2014). In the alternative view t, testing using a likelihood ratio or a chi-square statistics for the quasi-symmetric model and marginal homogeneity is carried out by comparing the original tallies with a set of values iteratively fitted to a log-linear model. In its simplest form, quasi-symmetry is a model for square contingency tables that states that certain odds ratios are symmetrical around the main diagonal. The model of quasi-symmetry requires that the expected marginal total of any one row of the table not the same as the expected marginal total for the corresponding column. A special case is complete symmetry, by which probabilities themselves are symmetric around the main diagonal.

Alternatively, quasi-symmetry will be considered purely as a mathematical property of a matrix. Specifically, any matrix whose entries are strictly positive, possess quasi-symmetry if it can be written as a product of a diagonal and a symmetric matrix. Once this has been proven, it will be shown that for any quasi-symmetric matrix, a direct inverse solution can be obtained for both the symmetric and diagonal matrix. In addition to definition of quasi-symmetry based without requiring knowledge of the limiting probabilities introduced by Caussinus (1965, p.147) and which has since become widely accepted for the analysis of contingency tables are illustrated in this study.

An important feature of the erodogic Markov chain is the property of reversibility generally found in cyclical lithologic transitions in stratigraphic succession (Davis, 2002). A direct relationship between reversible Markov processes and quasi-symmetry has been previously noted (McCullough, 1982; Richman and Sharp, 1990); the relationship is not easily understood until quasi-symmetry is considered as an explicit mathematical property. In other word, an erodogic Markov chain sequence is reversible if and only if the tally matrix of transition counts possesses quasi-symmetry or satisfies balance equation of Kolmogorov criterion. In the usual case where the tally matrix possesses marginal homogeneity, a simple chi-square test for symmetry is sufficient to determine the presence or absence of reversibility.

In this paper we illustrate the use of the quasi-symmetry model in the analysis of categorical data that summarized the constituents lithologic of cyclothems. The paper is organized as follows:

1) We recall some definitions and theorem about reversible Markov process and quasi-symmetry 2) followed by the illustration of real geological cyclothymiacs data examples and most likely their origin.

II. QUASI- SYMMETRY AND MARGINAL HOMOGENEITY

A. Quasi-Symmetry

Quasi-symmetry defined as the probability that an observation falls in *i* and *j* cell of a square matrix is not the same as the probability that it falls in *i, j* cell. The model of quasi-symmetry requires that the expected marginal total of any row of the matrix not the same as the expected marginal total for the corresponding column. A special case is complete symmetry, by which probabilities themselves are symmetric around the main diagonal.

Suppose A is an *n* x *n* matrix, that is, *n* rows and *n* column with $(i, j)^{th}$ entry f_{ij} is quasi-symmetric if and only if

$$f_{ij}\, f_{jk}\, f_{ki} = f_{ik}\, f_{kj}\, f_{ji} \quad \text{for any } i, j, k \quad \text{with } 1 \leq i, j, k \leq n \tag{1}$$

This definition of quasi-symmetry initiated by Caussinus (1965, p.147) and which has since become widely accepted for the analysis of contingency tables (see: Bishop et al., 1975, p.287; Agresti, 1990, p. 355; Sharp and Markham, 2000, p. 562). The above inequality can be simply stated as a Markov sequence is reversible if and only if the tally matrix transition counts possess quasi-symmetry and conversely Markov sequence is irreversible.

In the case where the tally transition counts matrix (f_{ij}) possess marginal homogeneity and the entries in the table are subject to random error, a simple Chi- square test is available for testing for complete symmetry $p_{ij} = p_{ji}$, $i \neq j$ is tested by calculating the test statistic. The applicable Chi-square test (Caussinus, 1965, p. 142; Plackett, 1981, p. 79; Sharp and Markham, 2000, p. 567) is given by:

$$\chi 2 = \sum_{i>j} \sum (f_{ij} - f_{ji})^2 / (f_{ij} + f_{ji}) \tag{2}$$

This is asymptotically distributed as $\chi 2$ with n (n-1) /2 degrees of freedom. Following the convention of Fisher, acceptance of the null hypothesis will be assumed at the 5% confidence level and rejection of the null hypothesis at the 1% confidence level.

III. Marginal Homogeneity

A transition count matrix *n* x *n*, has marginal homogeneity if the row sums match the column sums, i.e. $f_{i+} = f_{+i}$ for all *i*. In fact marginal homogeneity must hold provided that the succession of sedimentary units forms a continuum in which each unit is accessible from at least one other unit, an erodogic process. This is a condition which is commonly encountered because most sedimentary sections are counted during detailed stratigraphic traverses or from long continuous lithological profiles.

If there is any doubt, a Chi-square test ($\chi 2$) can be used to check for identity between row and column sums, that is:

$$\chi 2 = \sum (f_{i+} - f_{+i})^2 / (f_{i+} + f_{+i} - 2 f_{i+} f_{+i}) \tag{3}$$

This statistics asymptotically distributed, under the assumption of marginal independence, as $\chi 2$ with *n*-1 degrees of freedom. Acceptance of the null hypothesis at the 5 % confidence level is the same as rejection of quasi-symmetry.

If only transitions between litho-units have been counted, then the diagonal elements of the tally matrix are either absent or zero so that the test reduces to:

$$\chi 2 = \sum (f_{i+} - f_{+i})^2 / (f_{i+} + f_{+i}) \quad (4)$$

Stuart (1955) and Bhapkar (1966) were of the opinion that it is only approximation but Sharp and Markham (2000, p. 252) suggested that it would be adequate for most instances come across.

A. Geological Overview and Applications

To demonstrate the application of marginal homogeneity in geological data, an example has been taken from coal-bearing strata in the Bochumer Formation, Ruhrgebiet, Germany (Upper Carboniferous). Earlier workers (Fiebig, 1971 and many more) who studied these rocks have contended that these rocks consist of sedimentary successions and may be classified as cyclothems. Indeed, most of the early methods used in the study of cyclothems were essentially subjective in nature. Casshyap (1975) used simple mathematical and statistical model called a first order Markov chain model to determine lithologic transitions in vertical sequences (cyclicity) and compare the nature of cyclicity through space and time. The concept of cycles of sedimentation implies that the initial state (or lithology) determines to some degree the subsequent state (or lithology). Table 1 lists the tally count matrix of (A) siltstone, (B) shale, (C) sandstone, (D) coal, and (E) Rooty bed from Bochumer Formation (Westphal A-2) Ruhrgebiet, Germany (Casshyap, 1975, p. 243). Highest values of transition probability matrix (p_{ij}) link the lithologic states distinctly, and a strongly preferred upward transition path for litho-log changes that can be derived is: D → B → A → C → A → E → D suggesting symmetrical cycles. The depositional environment that can be obtained consisted mainly of high–constructive delta plain, in which distributaries and tributary channels and their sub-environments including natural levees and coal-forming swamps developed and migrated constantly across the plain to give rise interbedded symmetrical cyclic succession of Bochumer Formation.. In general, this order of lithologic transitions is closely comparable with that suggested for the Carboniferous coal measures of Scotland, and, likewise, fits suitably into the concept of deltaic cycles (Read, 1969).

TABLE 1

Tally Count Matrix of Bochumer Formation (f_{ij})

	A	B	C	D	E	f_{i+}
A	0	57	46	19	92	214
B	132	0	76	130	90	428
C	56	52	0	08	24	140
D	17	304	18	0	33	372
E	03	22	02	196	0	223
f_{+i}	208	435	142	353	239	1377

The inspection of the entries shows that the tally matrix possesses perfect marginal homogeneity indicating the litho-unit profiles have been properly counted. So a test for marginal homogeneity is not required. Observe first of all that the tally matrix lacks symmetry and this is confirmed by testing with of $\chi 2 = 294.54$ where $\chi 2$ (1%) = 23.21 with 10 degrees of freedom and is thus a case of quasi-symmetry. Consequently, the Bochumer Formation conforms to a non reversible Markov process, hence the succession possesses Markovian mechanism, and that the sequence represents cyclic sedimentation corroborating the inferences deduced by earlier workers on the basis of subjective approach.

Now consider a sedimentary sequence consists of three different litho-units of (A) sandstone, (B) shale, and (C) limestone from the Chester Series by Krumbein (Harbaugh and Bonham-Carter, 1970, p. 108). A traverse along such a sedimentary sequence of 309 counts will then yield the following tally matrix (Table 2).

TABLE 2

Tally Count Matrix Chester Series (f_{ij})

	A	B	C	f_{i+}
A	58	18	02	78
B	15	86	39	140
C	05	35	51	91
f_{+i}	78	139	92	309

As a properly counted sequence always possess marginal homogeneity, symmetry of the non diagonal entries in the tally matrix is sufficient to determine if the process conforms to that of a reversible Markov process and if there is any doubt, confirmation can be made using a simple Chi-square test as shown above. To be certain, a calculated test for symmetry (3) gives $\chi 2 = 1.64$ relative to a tabulated value of $\chi 2$ (1%) = 11.34 with v = 3. Consequently, the sedimentary sequence of Chester Series conforms to a reversible Markov process; hence the succession does not possess Markov cyclicity.

Earlier workers who studied the sedimentary strata of the Chester Series have asserted that these rocks consist of cyclic sedimentary succession (Weller, 1930; Heckle, 1986). Apart from giving elaborate differences in composition and thickness, these workers discussed their significance with respect to sedimentation and tectonics. Indeed, most of the early methods used in the study of cyclothems were essentially subjective in nature. As of today, more objective methods using mathematical and statistical models are available; their application is rapidly growing in the analysis of sedimentary successions following the pioneer work of Vistelius (1949). It was therefore considered to be highly appropriate that the problem of sediment cyclicity in the Chester Series should be re-examined quantitatively using marginal homogeneity criterion. As illustrated above, formal testing on a set of observed transition counts did not support the Weller's contention that the sequence being a cyclothymiacs. This possibly is apparently due to the fact that in naturally observed cyclothems, irregularities in transgressions and regressions obstruct in developing an ideal sequence.

In a another geological example the tally count matrix of four types of stratigraphic succession which occur in the Devonian rocks of Prince of Wales Island, Arctic Canada, an ancient alluvial plain succession (Miall , 1973, p. 351) consisting of (A) conglomerate,(B) pebbly sandstone,(C) coarse to medium sandstone, and (D) fine sandstone is considered (Table 3). Visual examination of the stratigraphic sections led Miall (1973, p. 356) to suggest that most of the fining upward cycles were originated by debris floods sweeping across the piedmont surface and creating new channels. Waning flow together with lateral accretion during normal periods deposited beds of successively finer grain size until the next flood altered the channel pattern once again.

TABLE 3

Transition Count Matrix of Devonian fluvial succession (f_{ij})

	A	B	C	D	f_{i+}
A	0	03	11	01	15
B	05	0	01	08	14
C	04	00	0	15	19
D	05	13	06	0	24
f_{+i}	14	16	18	23	72

It is observed that the tally matrix table has nearly perfect marginal homogeneity indicating the section has been properly counted. So a test for marginal homogeneity is not required. If the table is then tested for symmetry by using Chi-square statistics (3), the result gives $\chi 2 = 13.48$ where $\chi 2$ (5%) = 12.59 with 6 degrees of freedom. Thus, it can be concluded that this alluvial succession is a non-reversible Markov sequence; hence the succession possess Markov cyclicity. This statistical result further supported by Kolmogorov condition due to the existence of a non-symmetric zero entry in the position (3, 2) which has a non-zero in the (2, 3) position thereby the succession is non-reversible. Visual examination of the stratigraphic sections led Miall (1973, p. 356) to suggest that most of the fining-upward cycles originated by debris floods sweeping across the piedmont surface and creating new channels. Waning flow in addition to lateral accretion acting during periods of normal and quieter run-off gave way to deposit beds of successively finer grained sandstones, until the next flood episode altered the channel pattern once again so that once the flood has passed, the original fluvial pattern re-established itself. To sum up, the statistical study strongly support that the sediments succession was deposited by Markovian mechanism and as a whole represent fluvial sedimentation in predictable cyclic arrangements of litho-units.

IV. REVERSIBLE AND NON REVERSIBLE MARKOV PROCESS

A Markov sequence is a series of possible events developed by a Markov process in which the probability of changing from each event to another depends solely on the state attained in the previous event. Markov processes are among the most important of all random processes. In a sense they are the stochastic analogs in which the state of the system at time t_n is influenced by or dependent on the state of the system at time t_{n-1}, but not the previous history that led to the state at time t_{n-1}. The presence of an erodogic Markov process implies degree of order in a system, but it must be extended to a number of states so as to create a closed system to imply cyclicity. Markov chains have been used by several workers (*e.g.,* Doveton, 1994; Khan and Tewari 2007; Hota et al., 2012; Tita and Djomeni, 2016; Khan et al., 2020; and many more) to validate the presence of ordered and cyclic succession of lithofacies.

Ross (2019) defined a stochastic process (X_n, n = 0, 1, 2…) such that

$$P\,(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1} \ldots, X_1 = i_1, X_0 = i_0) = P\,(X_{n+1} = j \mid X_n = i) \qquad (5)$$

is an erodogic Markov chain.

Whereas Stroock (2014) is of the opinion that a class of Markov processes said to be reversible if, on every time interval, the distribution of the process is the same when it is run backward as when it is run forward i.e., for any $n \le 0$ and $(i_0, i_1 \ldots, i_n) \in S^{(n+1)}$, in the ***discrete time***,

$$P\,(X_m = i_m \text{ for } 0 \le m \le n) = P\,(X_{(n-m)} = i_m \text{ for } 0 \le m \le n) \qquad (6)$$

And in the ***continuous time,***

$$P\,[X\,(t_m) = i_m \text{ for } 0 \le m \le n] = P\,[X\,(t_n - t_m) = i_m \text{ for } 0 \le m \le n] \qquad (7)$$

Whenever $0 = t_0 \le \ldots \le t_n$. Indeed, depending on whether the setting is that of discrete or continuous time, we have

$$P\,(X_0 = i, X_n = j) = P\,(X_0 = j, X_n = i)$$

or

$$P\,[X\,(0) = i, X\,(t) = j] = P\,[X\,(t) = i, X\,(0) = j]$$

In fact, reversibility says that, depending on the setting, the joint distribution of (X_0, X_n) or [X (0), X (t)] is the same of (X_n, X_0) or [X (t), X (0)].

If P is the transition probability matrix and π_i is the limiting probability, then, by taking n = 1 in above equation, we see that

$$\pi_i\, p_{ij} = P\,(X_0 = I \cap X_1 = j) = P\,(X_0 = j \cap X_1 = i) = \pi_j\, p_{ji}$$

$$\boldsymbol{\pi_i p_{ij} = \pi_j p_{ji}} \quad \text{the condition of } \textbf{\textit{detailed balance}} \qquad (8)$$

That is, P satisfies that is a reversible and conversely, (8) implies non-reversibility.

The above discussion can be summarized in the form of a theorem.

Theorem 1: *A stationary process if and only if there exists a positive collection of number* (π_i) *to unity such that*

$$\pi_i\, p_{ij} = \pi_j\, p_{ji} \text{ for all } i, j \in S$$

whenever such a collection does not exists, the stationary Markov process is reversible.

Rubinstein and Kroese (2007) suggest the Markov transition probability that state i to state j and π_i and π_j are the equilibrium probabilities of being in state i and j respectively. Moreover the detailed balance equation (**$\pi_i\, p_{ij} = \pi_j\, p_{ji}$**) means that the probability of seeing a transition from state i to state j equals as seeing a transition from state j to i. A transition from *i to j for* the original Markov chain is a transition from *j to i* for the non-reversible Markov chain. In simple words, for each pair of state *i and j* the long run rate at which the chain makes a transition from state i to state j equals the long -run rate at which the chain makes a transition from state j to state *i.* Thus, the detailed balance equation (8) derived above allows us to determine if a markov process is reversible or irreversible based on the transition probability matrix and the limiting probabilities.

Theorem 2: *A Markov chain with transition matrix P is reversible if* $\Pi * P$ *is symmetric where* $*$ *means component-wise multiplication.*

Proof: Let R = $\Pi * P$

$$\Pi = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \quad * \quad P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

$$\Pi * P = \begin{pmatrix} \pi_1 p_{11} & \pi_1 p_{12} & \pi_1 p_{13} \\ \pi_2 p_{21} & \pi_2 p_{22} & \pi_2 p_{23} \\ \pi_3 p_{31} & \pi_3 p_{32} & \pi_3 p_{33} \end{pmatrix}$$

The Π * P matrix is symmetric matrix *i.e.* $\pi_i p_{ij} = \pi_j p_{ji}$ for all i, j, then the balance equations are satisfied. Thus the transition matrix P is ***reversible*** otherwise, P is ***non - reversible i.e***. the calculated transition matrix is non-symmetrical.

A. Geological Overview and Application

To illustrate the geological use of detailed balance equation (8) an example is taken from the coal measures of Lower Gondwana coalfield of peninsular India. Coal measure cyclothems or fluvial fining-upward cycles are good examples of sedimentary succession laid down under the control of Markovian processes. It is possible to describe the process statistically and once this is done, to interpret the results in considerable detail in terms of the evolution through time of the depositional mechanisms occurring in the given sedimentary basin. Tewari et al., (2009) applied Markov chain and entropy analysis to the Barakar Formation of Bellampalli, Andhra Pradesh, India, a formation of Early Paleozoic age, comprising a spectrum of sedimentary facies, including gritty sandstone, sandstone, siltstone, shale including carbonaceous and coal. The majority of cycles are symmetrical (type-B of Hattori) but asymmetrical cycles are present as well. Deposition of these various lithofacies was controlled by lateral migration of stream channels in response to varying discharge and rate of deposition across the alluvial plain.

Table 4 is a geological example of the Transition count and Transition probability matrices from the Barakar Formation of Bellampalli coalfield, Pranhita Godavari Graben Gondwana basin, India (Tewari et al., 2009), comprising (A) coarse-medium sandstone, (B) Interbedded sequence of shale and sandstone (C) carbonaceous shale, and (D) coal illustrates the application of Theorem 1.

TABLE 4

Transition count matrix of Barakar Formation (f_{ij})

	A	B	C	D
A	0	08	113	35
B	08	0	03	01
C	78	03	0	65
D	59	00	79	0

Transition probability matrix of Barakar Formation (p_{ij})

	A	B	C	D
A	0	0.05	0.73	0.22
B	0.67	0	0.25	0.08
C	0.53	0.02	0	0.45
D	0.43	00	0.57	0

The limiting distribution vector of transition probability matrix (p_{ij}) is π = (0.332, 0.016, 0.400, and 0.252). Now we compute R = Π * P

Reversibility matrix of Barakar Formation (R)

	A	B	C	D
A	0	0.016	0.242	0.073
B	0.012	0	0.004	0.001
C	0.232	0.008	0	0.180
D	0.108	0	0.143	0

Inspection of the rows and columns shows that the table is nonsymmetrical. By Theorem 1 if the transition count matrix represents counts from a Markov sequence, then acceptance of no symmetry implies acceptance of Markov non-reversibility. To be certain, a chi-square test as given by Plackett, (1981) and applied by Sharp and Markham, (2000) as described above is used here, a calculated test for symmetry (3) for the above data gives $\chi2$ = 15.89 on 5 degrees of freedom which is large (Actually, n (n-1)/ 2 = 6, but because one pairs of frequencies equal zero, they were not counted). In fact, it is usually large because its falls above the lower 0.05 point on the ($\chi2$) Chi-squared distribution with 5 degrees of freedom ($\chi2$ =11.07). The matrix is non-symmetrical than could be due to chance hence this succession of the lithofacies is non-reversible Markov sequence, hence the succession possess Markov cyclicity. Thus the statistical results support the geological interpretation that the Barakar formation is a cyclothymiacs and is explained by the lateral migration of stream channels in response to varying discharges and rate of deposition across the alluvial plain. Factors related to tectonism may have controlled the channel pattern of depositing streams.

Theorem 3*: A transition matrix is not reversible if there exists any non-symmetric zero entry in the matrix.*

Proof: The proof is quite simple because the detailed equation or the Kolmogorov condition as described elsewhere is violated in such cases.

B. Geological Overview and Application:

As an example the Lower Cretaceous and Tertiary Formations of the Kombe-Nsepe are southeastern part of the Douala sub-basin of Cameroon (Tita and Djomeni, 2016) is selected to illustrate the test for Markov reversibility and a practical application of Theorem 2. These formations are the main hydrocarbons bearing sedimentary successions of the Douala sub-basin. The observed tally count and transition probability matrices are listed below in Table 5.

TABLE 5

Transition count matrix of Lower Cretaceous Formation (f_{ij})

	A	B	C	D	E
A	0	33	59	07	02
B	22	0	30	0	01
C	75	18	0	09	09
D	05	01	09	0	0
E	0	02	11	0	0

Transition probability matrix of Lower Cretaceous Formation (p_{ij})

	A	B	C	D	E
A	0	0.33	0.58	0.07	0.02
B	0.42	0	0.57	00	0.01
C	0.68	0.16	0	0.08	0.08
D	0.33	0.07	0.60	0	00
E	00	0.15	0.83	00	0

Where A- Dark grey shale, B- Interbedded fine grained sandstone and shale, C- Medium to coarse grained sandstone, D- Coarse grained sandstone, and E- Carbonate/ argillaceous shale.

Tita and Djomeni (2016) have made a detailed study of the lithofacies and paleo-environmental evolution of the Early Cretaceous to Tertiary series which are common in West Africa basins and identified five lithofacies (states) viz. Dark/grey shale, interbedded fine grained sandstone, conglomerate facies and carbonate and argillaceous shale. The lithofacies sequence and association suggest that the study area formed a part of a northward-flowing fluvial system. Statistical analysis of the lithofacies sequence reveals cyclicity (fining upward) and prefentially asymmetrical cyclical deposition of the post-rift Cretaceous and Tertiary Formations. Auto-cyclic causes, such as in-channel, point bar, crevassing and stream avulsion generated the cyclic deposit within a framework of fluvial-alluvial environments.

Evidently, the transition probability table is non-symmetric and therefore the (p_{ij}) is not reversible since there are zero (00) entries in the position (2, 4), (4, 5) and (5, 1) and (5, 4) positions which has a non- zero in the (4, 2), (5, 4), (1, 5) and (4, 5) positions. Consequently, the sedimentary sequence conforms to a reversible Markov process; hence the sedimentary sequence does not possess Markov cyclicity. As illustrated above (theorem 2), formal testing on a set of observed transition counts did not support the suggestion that the lower Cretaceous formation, Douala sub-basin is a cyclothymiacs. To verify and strengthen above conclusion the chi-square ($\chi 2$) statistics is calculated for symmetry (3) with a $\chi 2$ = 17.06 relative to a tabulated value of $\chi 2$ (1%) = 23.21 with 10 degrees of freedom. The sedimentary sequence of Douala sub-basin, thus, conforms to a reversible Markov process and hence formal testing on a set of observed transition counts did not support cyclothymiacs sequence and hence the conclusions drawn by Tita and Djomeni (2016) that the observed asymmetrical cyclothems of the post-rift Cretaceous and Tertiary Formations are not justified.

Definition 1: An n x n stationary Markov chain matrix with *(i, j)* th entry p_{ij} is reversible if and only if, any path starting from state i and back to i has the same probability as the path going in the reverse direction. This is the definition of reversibility introduced by Andrei Kolmogorov and known as Kolmogorov criterion. Kolmogorov gives a necessary and sufficient condition for a Markov chain to be reversible directly from the transition matrix probabilities. The criterion requires that the products of probabilities around the loop i to j to l to k returning to i must be equal *i.e.*, transition matrix P is reversible if and only if its transition probabilities satisfy:

$$p_{ij}\, p_{jl}\, p_{lk}\, p_{ki} = p_{ik}\, p_{kl}\, p_{lj}\, p_{ji}$$

C. Geological Overview and Example:

A practical application of the Kolmogorov criterion is illustrated by Le- Roux (1992, p. 176) for alluvial sediments for which a transition count matrix of five sandstone lithofacies namely Mudstone-clast conglomerate (A), structure-less sandstone (B), plane-bedded sandstone (C), cross-bedded sandstone (D) and mudstone (E) within the Banksgaten sandstone forming the Beaufort Group of the Karoo basin, South Africa (Table 6).

TABLE 6

Transition count matrix of Beaufort Group (f_{ij})

	A	B	C	D	E
A	0	39	05	21	03
B	26	0	28	27	01
C	01	30	0	06	04
D	10	26	16	0	06
E	01	04	00	03	0

A= Mudstone-clast conglomerate, B = structure-less sandstone, C = plane-bedded sandstone, D = cross-bedded sandstone, E = mudstone

Transition probability matrix of Beaufort Group (p_{ij})

	A	B	C	D	E
A	0	0.573	0.073	0.309	0.044
B	0.317	0	0.341	0.329	0.012
C	0.024	0.731	0	0.146	0.097
D	0.172	0.448	0.276	0	0.103
E	0.125	0.500	00	0.375	0

Thus, we have

$p_{ij}p_{jl}p_{lk}p_{ki}$ = 0.573x0.341x0.146x0.103x0.125 $p_{ik}p_{kl}p_{lj}p_{ji}$ = 0.375x0.276x0.731x0.317x0.044

For a 5 x 5 transition probability matrix, it turns out that only one detailed balance equation has to be checked and found to be non-equal suggesting that P is non-reversible which in turn supports the geological interpretation of cyclical deposition within the Banksgaten sandstone Karoo Group as discussed in detail by Le Roux (1992).

Another assessment made by using Theorem 2 That '*A transition probability matrix is not reversible if there exists any non-symmetric zero entry in the matrix'*. According to this $P_{i\,j}$ is not reversible since there is a 00 entry in the (5, 3) position has a non-zero in the (3, 5) position further strengthen the Le-Roux's geological interpretation.

Theorem 4**:** *A stationary Markov chain is reversible if and only if the matrix of transition probabilities can be written as the product of a symmetric and a diagonal matrix i.e.* P *is the observed transition probability matrix for a reversible Markov chain if and only if* P = SD *where* S *is a symmetric matrix and* D *a diagonal matrix such that* $\sum_j p_{ij} = 1$

Proof: Let P be the transition matrix of a reversible Markov chain. We will show that P can be written as a product of a symmetric matrix (S) and a diagonal matrix (D) *i.e.*, P = SD, where S is an *n x n* symmetric matrix whose (*i, j*) th entry is s_{ij} and D is a diagonal matrix whose *i* th diagonal entry is d_i . Since $p_{ij} = s_{ij}\, d_{ij}$ and $s_{ij} = s_{ji}$ for all *i, j* .

$$\text{Let } P = SD = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$$

$$\begin{pmatrix} s_{11}d_{11} & s_{12}d_{22} & s_{13}d_{33} \\ s_{21}d_{11} & s_{22}d_{22} & s_{23}d_{33} \\ s_{31}d_{11} & s_{32}d_{22} & s_{33}d_{33} \end{pmatrix}$$

Multiply both sides by (d_{11}, d_{22}, d_{33} …); then we have

(d_{11}, d_{22}, d_{33} …)

$$P = (d_{11}, d_{22}, d_{33} \ldots) \begin{pmatrix} s_{11}d_{11} & s_{12}d_{22} & s_{13}d_{33} \\ s_{21}d_{11} & s_{22}d_{22} & s_{23}d_{33} \\ s_{31}d_{11} & s_{32}d_{22} & s_{33}d_{33} \end{pmatrix}$$

on multiplication we get

$= (s_{11}d_{11}\, d_{11} + s_{21}d_{22}\, d_{11} + s_{31}d_{33}\, d_{11} + \ldots\ldots, s_{12}d_{11}\, d_{22} + s_{22}d_{22}\, d_{22} + s_{32}d_{33}\, d_{22} + \ldots\ldots,$

$= [d_{11}\,(s_{11}d_{11} + s_{21}d_{22} + s_{31}d_{33}) + \ldots\ldots,\ d_{22}\,(s_{12}d_{11} + s_{22}d_{22} + s_{32}d_{33}) + \ldots\ldots,)\ ]$

$= [d_{11}(s_{11}d_{11} + s_{12}d_{22} + s_{13}d_{33}) + \ldots\ldots,\ d_{22}\,(s_{21}d_{11} + s_{22}d_{22} + s_{23}d_{33}) + \ldots\ldots,)\ ]$

$= [d_{11}\sum_j p_{i\,j},\ d_{22}\sum_j p_{i\,j},\ d_{33}\sum p_{i\,j} \quad \ldots\ldots]$ Since $\sum_j p_{ij} = 1$ the above identity can be written as

$= (d_{11}, d_{22}, d_{33} \ldots)$

Thus we have

$(d_{11}, d_{22}, d_{33} \ldots)$ P = $(d_{11}, d_{22}, d_{33} \ldots)$

We can rewrite this as

$1/\sum_k d_{kk}\ (d_{11}, d_{22}, d_{33} \ldots)\ P = 1/\sum_k d_{kk}\ (d_{11}, d_{22}, d_{33} \ldots)$

From this we can calculate stationary distribution π as

$\pi = [d_{11}/\sum_k d_{kk}\ , d_{22}/\sum_k d_{kk}\ , d_{33}/\sum_k d_{kk}\ , \ldots]$

$\pi\ p_{ij} = d_{ii} / \sum_k d_{kk}\ s_{i\,j}\, d_{j\,j}$

since $(p_{ij}) = (s_{ij}\, d_{jj})$ and $s_{i\,j} = s_{j\,i}$ for all i, j as assume in proof, we get

$\pi_j\ p_{j\,i} = d_{j\,j} / \sum_k d_{kk}\ s_{j\,i} d_{i\,i}\ =\ \pi_i\ p_{i\,j}$

Thus we get the detailed balance equation (Kolmogorov condition) for the Markov chain defined by P = SD, therefore the Markov chain is reversible.

D. Geological Overview and Example

The following examples illustrate how Theorem 4 can be applied to determine reversibility in geological studies. Vistelius and Faas (1973) observed grain sequences in Pacolet Mills Pluton in South Carolina. The Pacolet Mills Pluton as described by Secor et al., (1983) is a fine to medium grained metamorphosed granite, which appears in places to be intermixed with biotitic gneiss. Fracturing and shearing possibly associated with metamorphism, have allowed weathering of the granite. Vistelius (1987, p. 592) described an important feature of the texture of Pacolet granite, an observed sequence of microcline, plagioclase, quartz and biotite should be a property of Markov reversibility *i.e.*, along any linear traverse through a specimen of granite, there will be no statistical difference in the number of observed grain transitions between the forward and the reverse direction. This develop for an important distinction between reversible Markov grain transitions in granite and the generally nonreversible or cyclical lithologic transitions found in stratigraphic sections (Davis, 2002).The following Transition count matrix (Table 7) as observed in grain sequence in Pacolet Mills granite which consist of (M) microcline, (P) plagioclase, (Q) quartz and (B) biotite:

TABLE 7

Transition Count Matrix (N)

	M	P	Q	B
M	20	42	47	25
P	53	27	47	23
Q	40	52	58	32
B	25	26	26	16

Replacement Matrix (N) to Symmetric Transition Count Matrix (S)

	M	P	Q	B
M	20	47	43	24
P	47	27	52	24
Q	43	52	58	29
B	24	24	29	16

Transitional Probability Matrix (P)

	M	P	Q	B
M	0.149	0.350	0.321	0.179
P	0.313	0.180	0.346	0.160
Q	0.236	0.286	0.318	0.159
B	0.258	0.258	0.312	0.172

Stationary probabilities (π) or diagonal elements of diagonal matrix (D)

(d_{11}, d_{22}, d_{33}, d_{44}) *i.e.* (M =0.2397, P=0.2683, Q=0.3256, B=0.1664) then we have

Reversibility matrix ($S= PD^{-1}$)

	M	P	Q	B
M	0.623	1.307	0.985	1.077
P	1.307	0.671	1.065	0.961
Q	0.985	1.065	0.979	0.958
B	1.077	0.961	0.958	1.034

By inspection it can be seen that S is symmetrical and fulfilling Kolmogorov criterion ($\pi_j\, p_{ji} = \pi_i\, p_{ij}$) suggesting thereby that the Markov process is reversible and, hence, the Pacolet Mills Pluton (South Carolina) had reversible grain sequence. This conclusion gets support Vistelius (1972) visual observation that ideal granite should be a reversible Markov sequence.

Another way to compute and proof Theorem 4 as follows

Let P is the observed transition probability matrix of a reversible Markov chain. We will show that P can be written as a product of a symmetric matrix (S) and a diagonal matrix (D) whose diagonal elements are the stationary probabilities of P. *i.e.*, P= SD

Let D = $\begin{pmatrix} \pi_1 & 0 & 0 \\ 0 & \pi_2 & 0 \\ 0 & 0 & \pi_3 \end{pmatrix}$ and P = $\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$

Then according to above theorem $S = P\ D^{-1}$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} 1/\pi_1 & 0 & 0 \\ 0 & 1/\pi_2 & 0 \\ 0 & 0 & 1/\pi_3 \end{pmatrix}$$

$$\begin{pmatrix} p_{11}/\pi_1 & p_{12}/\pi_2 & p_{13}/\pi_3 \\ p_{21}/\pi_1 & p_{22}/\pi_2 & p_{23}/\pi_3 \\ p_{31}/\pi_1 & p_{32}/\pi_2 & p_{33}/\pi_3 \end{pmatrix}$$

As we have proved above that if $\pi_i p_{ij} = \pi_j p_{ji}$, then the Markov chain is reversible. Thus we get

$$S = \begin{pmatrix} p_{11}/\pi_1 & p_{21}/\pi_2 & p_{31}/\pi_3 \\ p_{21}/\pi_1 & p_{22}/\pi_2 & p_{32}/\pi_2 \\ p_{31}/\pi_1 & p_{32}/\pi_2 & p_{33}/\pi_3 \end{pmatrix}$$

This is clearly a symmetric matrix. Since S = PD^{-1}, we get P = S D, as required.

E. Geological Overview and Application:

Strata of the Miocene-Pliocene Middle Siwalik molasses of Kaluchaur area in Uttrakhand exposed in excellent road-cuts and along river course, offer a good opportunity to study alluvial plain cyclicity and relate it to recurring patterns of depositional facies sequence. Early workers studying the Middle Siwalik concentrated on the vertebrate fossils and little published information exists on characteristics of the deposits with respect to the nature of cyclicity and facies variation. Moreover, the cyclic aspects of Middle Siwalik sediments have been based only upon visual appraisal of outcrops (Agarwal and Singh, 1983; Kumar et al., 2004) and more recently (Kundu et al., 2012; Kotlia et al., 2016). The exposed sequence consists of a spectrum of near horizontal depositional units representing a non-marine; cyclic, fluvial dominated lithofacies sequence shows a succession of repetitive lithofacies. Four major lithofacies are readily identifiable in the field, and, are distinctive and genetically meaningful are considered for partial independence and entropy analysis. The fining-upward asymmetrical sequence involves upward point bar sandstones and associated top stratum deposits or alternatively consists of levee-splay silt and flood plain variegated mud (Khan, 1996). Entropy plots of each lithological state corresponds to the Type A-4 category, signifying lower and upper truncated asymmetrical cycles and falls within the boundary allocated for fluvial-alluvial succession. These truncated fining-upward sequences are produced by lateral accretion in the channel belt environment, or crevassing and subsequent abandonment of crevasse channels leading to silting by levee and flood plains fines (Galloway, 1981). Data in the transition count matrix are then processed into transition probability matrix are given in Table 8

TABLE 8

Transition count matrix of Siwalik molasses (f_{ij})

	COSD	FMSD	SLSD	MDST	f_{i+}
COSD	0	28	05	02	35
FMSD	17	0	23	05	45
SLSD	10	07	0	14	09
MDST	07	10	04	0	21
f_{+i}	34	45	32	21	132

Note that the table has perfect marginal homogeneity i.e., $f_{i+} = f_{+i}$ indicating a properly counted section.

Transition probability matrix of Siwalik molasses (p_{ij})

	COSD	FMSD	SLSD	MDST
COSD	0	0.800	0.142	0.057
FMSD	0.377	0	0.511	0.111
SLSD	0.322	0.225	0	0.452
MDST	0.333	0.476	0.190	0

No we calculate out its limiting probability as

$\pi = (0.3460, 0.3183, 0.1972, 0.1384)$

Let D be a diagonal matrix with (π_i) as its diagonal entries. Then

$$S = PD^{-1}$$

$$\begin{pmatrix} 0 & 0.800 & 0.142 & 0.057 \\ 0.377 & 0 & 0.511 & 0.111 \\ 0.322 & 0.225 & 0 & 0.452 \\ 0.333 & 0.476 & 0.190 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1/0.3460 & 0 & 0 & 0 \\ 0 & 1/0.3183 & 0 & 0 \\ 0 & 0 & 1/0.1972 & 0 \\ 0 & 0 & 0 & 1/0.1384 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2.5157 & 0.7258 & 0.3626 \\ 1.0693 & 0 & 2.5888 & 0.7971 \\ 0.9248 & 0.6918 & 0 & 3.2608 \\ 0.9537 & 1.5094 & 0.9644 & 0 \end{pmatrix}$$

The above result shows that S is a non-symmetric matrix hence P is a non reversible matrix suggesting that the succession possess Markov cyclicity supporting Khan's geological interpretation that the Middle Siwalik molasses' is a cyclothymiacs in nature.

The following proof of the above Theorem 4 taken from Sharp and Markham (2000) based on transition count matrix as follows:

Let assume that A = DS, where S is an n x n symmetric matrix whose *(i, j) th* entry is s_{ij} and D is a diagonal matrix whose *i th* diagonal entry d_i. Since $f_{ij} = d_i s_{ij}$ and $s_{ij} = s_{ji}$ for all i, j, then

$$\begin{aligned} f_{ij} f_{jk} f_{ki} &= d_i s_{ij} d_j s_{jk} d_k s_{ki} \\ &= d_i d_j d_k s_{ij} s_{jk} s_{ki} \\ &= d_i d_j d_k s_{ji} s_{kj} s_{ik} \\ &= d_j s_{ji} d_k s_{kj} d_i s_{ik} \\ &= f_{ji} f_{kj} f_{ik} \\ &= f_{ik} f_{kj} f_{ji} \qquad \text{for all } i, j, k \end{aligned}$$

Hence A is quasi-symmetric following the definition given by Caussinus (1965, p. 147) and it can be written as a product of a diagonal and symmetric matrix. In other words a Markov process is reversible if the tally matrix of transition counts possess quasi-symmetry hence cyclical otherwise non-cyclical. An n x n matrix A with *(i, j) th* entry f_{ij} is quasi-symmetric if

$$f_{ij} f_{jk} f_{ki} = f_{ik} f_{kj} f_{ji} \qquad \text{for any } i, j, k \text{ with } 1 \le i, j, k \le n \qquad (9)$$

Now if the transition count matrix A is a quasi-symmetric matrix then by equation (1), we have

$(fij/f_{ji})f_{jk} = (f_{ik}/f_{ki})f_{kj}$ for all $1 \le i, j, k \le n$ (10)

Suppose Q_i be a diagonal matrix whose *j th* diagonal entry is f_{ij}/f_{ji}, for j = 1, 2, 3, … , *n* or a diagonal matrix whose *k th* diagonal entry is f_{ik}/f_{ki} for *k*= 1, 2, ,3, …..,*n* and **S** an *n* x *n* tally symmetric matrix whose (*j*, *k*) *th* entry is s_{jk}. Now assume Q_i A = S**,** then s_{jk}= $(f_{ij}/f_{ji})f_{jk}$ and $s_{ki} = (f_{ik}/f_{ki})f_{kj}$ .By equation (9)as obtained above, we have $s_{jk} = s_{ki}$; hence **S** must be symmetric. It has been shown previously (Richman and Sharp, 1990, p.754) suggested that Q_i A is symmetric for all *i* in (1,2,….,*n*). The simple way of doing this is by forming product Q_i A for all *i* where the entries in the diagonal matrix Q_i are obtained from the ratios of (f_{ij}/f_{ji}) of the quasi-symmetric matrix. To have unique symmetric matrix, each the un-scaled matrices can be multiplied by a constant k_i, obtained by taking the ratio of the sum of non-diagonal entries of observed tally matrix to the sum of the non-diagonal entries of each Q_i A (Sharp and Markham, 2000). Then a scaled estimate of S_i is given by k_1Q_1A and all k_1Q_1A have perfect symmetry in absence of random error. Sharp and Markham (2000, p. 568) suggests that in principle, if A (tally matrix) possess quasi-symmetry, then all scaled solutions of the square matrix should be identical and pass a chi-square test of symmetry as follows:

$$\chi 2 = \sum_{i>j} \sum (s_{ij} - s_{ji})^2 / (s_{ij} + s_{ji})$$

with $v = (n-1)(n-2)/2$ degree of freedom. s_{ij} denote the (i, j) *th* entry of the product matrix $k_1Q_1A = S_i$

F. *Geological Overview and Application:*

In a practical geological application of the Sharp and Markham's (2000) proof, a set of tallies from Late Paleozoic Barakar coal measures, East Bokaro basin, India (Khan and Casshyap, 1981, p.157) is taken (Table 8), which comprises A - sandstone, B- arenaceous shale, C - carbonaceous shale, and D - coal. These sediments are ideally deposited in the sequence A B C D A B …. C D and as a whole represent fluvial sedimentation in predictable cyclic arrangements of litho units.

TABLE 9

Observed tally matrix of Barakar coal measures (A)

	A	B	C	D	*Total*
A	0	1274	277	93	1644
B	1014	0	716	168	1898
C	500	628	0	556	1648
D	154	249	309	0	712
Total	1668	2151	1302	817	0

Note the absence of symmetry and marginal homogeneity

Test of symmetry $\chi 2 = 185.364$ $v = 6$ $\chi 2$ (1%) = 16.81

Marginal homogeneity $\chi 2 = 72.463$ $v = 3$ $\chi 2$ (1%) =11.34

Formal testing ($\chi 2$) confirms the rejection of symmetry and marginal homogeneity

Scaled inverse square matrices k_1Q_1A				**Diagonal matrices** $(k_1Q_1)^{-1}$	**Test for quasi-symmetry**
0	1400.63	304.53	102.24	0.9095	$k_1 = 1.0994$
1400.63	0	988.99	232.05	0.7240	

304.53	382.27	0	338.63	1.6420	
102.24	165.28	205.11	0	1.5064	$\chi 2$ = 311.79

k_2Q_2A				$(k_2Q_2)^{-1}$	
0	957.43	205.15	69.87	1.3308	k_1 =0.9442
1400.63	0	676.04	108.22	1.0591	
538.24	676.04	0	598.51	0.9290	
98.08	158.59	196.81	0	1.4912	$\chi 2$ = 362.22

k_3Q_3A				$(k_3Q_3)^{-1}$	
0	2053.01	427.85	143.57	0.6476	k_1 =0.8556
760.83	0	537.24	126.02	1.3328	
427.80	537.31	0	475.71	1.1687	
115.50	186.77	231.85	0	0.6496	$\chi 2$ = 692.23

k_4Q_4A				$(k_4Q_4)^{-1}$	
0	1739.53	379.09	69.87	0.7302	k_1 =0.9442
1242.73	0	877.52	108.22	0.8159	
229.49	288.20	0	598.51	2.1279	
127.35	205.92	255.54	0	1.2092	$\chi 2$ = 362.22

Reject quasi-symmetry because $\chi 2$ (1%) = 11.34 with v =3

Each complete cycle of the Barakar sandstone start with conglomerate or coarse grained sandstone at the base, succeeded in turn, by parallel laminated siltstone-shale and terminates with shaly coal/ coal. These cycles are comparable to each other at various localities in the East Bokaro basin and show fining-upward character, which might have resulted from variations in hydraulics changes of lateral shifting of the braided/meander belts as described by Tewari and Khan (2017). Repeated occurrence of the sandstone-siltstone-shale facies suggests that channel establishment and abandonment was repeated many times at a given site upward facies transition is non-stationary over the entire areas and develop either locally in a part of the floodplain or occupy the entire basin overlying the deposits of various sub-environments. Above result support that the coal measures of the Barakar succession was deposited by Markovian mechanism and as a whole represents fluvial sedimentation deposited in a predictable cyclic arrangements of lithofacies.

Note that observed tally matrix has lacks both symmetry and marginal homogeneity and formal testing by equations (2) and (3) confirm these result with $\chi2$ = 185.364 v = 6; $\chi2$ (1%) = 16.81 and $\chi2$ = 72.463 v = 3; $\chi2$ (1%) =11.34. Such a tally matrix of Late Paleozoic Cyclical coal measures, East Bokaro basin, India is a case for quasi-symmetry (Table 9).

An application of chi-square tests to each of the scaled inverse square matrices shows that they all lack symmetry and consequently the observed tally matrix does not possess quasi-symmetry After considering several possibilities, the statistical analyses evidently suggest:

1. Since the table lacks marginal homogeneity, the collection of data from a series of drill holes may possibly have a sampling bias in the observed transition counts.

2. None of the four inverted square matrices possess symmetry (Table 9), the observed tally matrix, therefore, cannot be quasi-symmetric and the sedimentary sequence must conform to a non-reversible Markov process.

Because the reverse and forward sequence are distinctly different, the bedding sequence from the Late Paleozoic Barakar coal measures, East Bokaro basin, India can be used to distinguish whether or not the section is overturned. That is, the geological interpretation of cyclical deposit within the East Bokaro basin as illustrated in Khan and Casshyap (1981) is supported unreservedly.

V. CONCLUSIONS

A Markov sequence is a series of states generated by a Markov process in which the probability of changing from one to another depends solely on the state which the system is in. If transition counts are made and tabulated following an ergodic process among discrete lithology, the corresponding row and column sums are equal or nearly equal *i.e*., $f_{i+}=f_{+}\, i$ all values of i. Such transition count matrixes have marginal homogeneity and are either symmetric or non-symmetric.

If the transition tally matrix has marginal homogeneity, a simple chi-square test for symmetry is sufficient to show that the sedimentary succession follows a reversible or a non-reversible Markov sequence. If the sequence is reversible, then the sedimentary succession lacks Markov cyclicity and on the other hand if the sequence in non-reversible then the succession possesses Markov cyclicity. Clearly if the sequence of lithological states is cyclical then the sequence of states cannot be reversible. The classical case of a cyclical sedimentary succession is the cyclothymiacs. Alternatively if the tally matrix lack marginal homogeneity and likely origin is a sampling bias was introduced by the counting procedure of the sequence, even then it is still be possible to recognize that the reversible Markov sequence by demonstrating that the tally matrix possesses quasi-symmetry. Thus a simple chi-square test for symmetry on the original tally matrix has immediate application in determining whether or not a sedimentary succession follows a reversible or non-reversible Markov process.

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