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Consumption and Non-Life Insurance Demand With Ambiguous Mortality

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Abstract **-** *This study considers the optimal consumption and non-life insurance demand for an individual who concerns about the ambiguous hazard rate of her/his lifetime. We formulate the ambiguity of the mortality in terms of Girsanov's probability measure changing by which the hazard rate changes with respect to the equivalent measures. For two cases without and with the consideration of the ambiguity aversion in the mortality, we respectively obtain the closed-form solutions for the value function, the robust consumption policy, and non-life insurance demand for the individual by using the technique of dynamic programming principle. Our findings can be concluded as follows. The ambiguity about the lifetime does not affect the non-life insurance demand policy because of independence between mortality risk and wealth risk. The insurance demand is increasing with respect to risk-free interest rate, the individual's wealth, age, and the maximum wealth loss, but decreasing with respect to the frequency of the wealth loss and the price of insurance. As for consumption, the individual saves more and consumes less at youth, and consumption is increasing between middle age and retirement, which is completely consistent with the people's behavior. The price of insurance has little impact on consumption, while the ambiguity about life has a significant impact on consumption. The more uncertain the reference model, the more the individual would like to consume.*

Keywords - *ambiguous hazard rate; Robust consumption policy; HJB-equation; insurance demand*

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I. INTRODUCTION

For an individual, the problem that how to avoid property loss and how to make a consumption plan for a future life is very important. Intuitively, the risk of an individual's residual lifetime should play the main role in impacting the consumption plan and insurance demand. In fact, some researchers have studied the optimal insurance and consumption policies with considering the mortality risk. For example, Yarri (1964) investigated the various aspects of consumer allocation and followed with interest the consumer's bequests. Yaari (1965) studied the optimal consumption and life insurance policies for an individual under uncertain lifetime, becoming the derivation of the "annuity puzzle". Pliska and Ye (2007) considered optimal life insurance and consumption policies for a wage earner whose lifetime is random and obtained the closed-form solutions of optimal life insurance and consumption policies. Besides investigating the life insurance policy, many researchers also studied the non-life insurance demand. For example, Briys (1986) used the methods of Merton (1969) to analyze the optimal consumption and non-life insurance policies in a continuous-time setting and provided a framework for the research of the non-life insurance decision problems. As an extension to Briys's work, Moore and Young (2006) considered the optimal non-life insurance, investment, and consumption policies for an individual who is concerned with his/her random lifetime.

We note that the above studies assumed that the lifetime of an individual is uncertain, while they also assumed that the individual has a clear knowledge of the probability of mortality, in which the hazard rate of mortality was assumed to be given. However, as the study of, an individual can not be fully aware of the real hazard rate of mortality, and most decisions are made based on her/his subjective beliefs. In other words, the individual is ambiguous about the hazard rate of mortality. Many researchers have focused on the existence of ambiguities in the financial market and the insurance market, such as Uppal and Wang (2003), Maenhout (2004) and Liu et al. (2021), etc. So far, only a few researchers have studied individuals' economic behavior under ambiguous hazard rates of mortality. For example, Young and Zhang (2016) considered the optimal investment policy for an individual who seeks to minimize the probability of lifetime ruin when his/her hazard rate of mortality is ambiguous. In their work, they assumed that the consumption rate is given by a constant. In contrast to the literature mentioned above, we allow an individual not only to have an ambiguous hazard rate but also to determine the consumption rate and non-life insurance policies.

In this paper, we consider the optimal property insurance and consumption policies for an individual who seeks to maximize his/her expected utility of terminal wealth and expected utility of consumption over the death/retirement time subject to ambiguous hazard rate. We focus on the robust consumption policy and non-life insurance demand for an individual. The individual can buy non-life insurance against potential losses of property. The financial ingredients of the wage earner's wealth include a risk-free asset, wages, consumption, and non-life insurance. The remainder of this paper is organized as follows. We formulate our model and solve the optimal consumption policy and non-life insurance demand without ambiguity as a benchmark in Section 2. In Section 3, we formulate the ambiguity of the mortality in terms of Girsanov's Theorem and probability measure changing. The closed-form solutions for the value function, the robust consumption policy, and non-life insurance demand are obtained for the individual by using the technique of dynamic programming principles. We also compare the optimal consumption policies between the cases of constant hazard rate and ambiguous hazard rate. Finally, several numerical examples and economic explanations are given to illustrate our results.

II. Model setup and benchmark

We start with a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$, in which all random variables in the rest of the paper are well-defined. For an individual, let $W(t)$ be her/his wealth at the time t and denote $0 \le c(t) < \infty$ as the consumption rate at the time t. Assume that the individual can save the wealth in a bank account with a constant risk-free interest rate $r \ge 0$. Let the random variable ι be the random death time of the individual. To make the problem more tractable, we follow a common assumption that ι is an exponential random variable with a constant hazard rate λ (a similar assumption can be seen in Bayraktar and Zhang (2015) and Young and Zhang (2016)). In addition, we assume that the individual earns a wage at a rate $I(t)$ during the period $[0, T\wedge\iota]$, where $T < \infty$ serves as the time of retirement and $T\wedge\iota := \min\{\iota, T\}$. Without losing generality, we assume that the function $I:[0,T]\to R^+$ satisfies $\int_0^T I(s)\mathrm{d}s < \infty$.

What's more, the individual may be confronted with potential losses of property which can be modeled as a compound Poisson process, which $N(t)$ is a Poisson process with jump intensity $\kappa > 0$. For each loss, the maximum loss amount is proportional to the current wealth with $0 < \nu \le 1$, that is $\nu W(t)$. Then, the wealth of the individual without insurance can be described by the following process

$$\mathrm{d}W(t) = (rW(t) - c(t) + I(t))\mathrm{d}t - \nu W(t)\mathrm{d}N(t). \tag{2.1}$$

To avoid the huge loss of wealth, the individual usually transfers this risk by buying non-life insurance coverage in practice. With insurance with a given proportional coverage, the wealth process becomes

$$\mathrm{d}W(t) = (rW(t) - c(t) + I(t) - \varpi(\beta(t)))\mathrm{d}t + [\beta(t)-1]\nu W(t)\mathrm{d}N(t), \tag{2.2}$$

Where $0 \le \beta(t) \le 1$ is the proportional degree of coverage covered by the insurer at the time t and $\varpi(\beta(t))$ represents the corresponding premium rate charged by the insurer? Assume the insurer provided the coverage calculates the premium in terms of expected value principle, and the safety loading is $\theta > 0$, then, $\varpi(\beta(t)) = (1+\theta)\kappa\beta(t)\nu W(t)$.

From the perspective of economics, decisions making is usually based on the expected utilities in the future. For this individual, we assume that she/he seeks to maximize the expectation of discounted utilities from both the future consumptions and the terminal wealth at the retirement time if he/she is alive. In the wealth model, both $c(t)$ and $\beta(t)$ are served as control variables. Therefore, we define the value function as follows:

$$V(t,w) = \sup_{c(t),\beta(t)} E_{t,w}\left[\int_t^{\iota\wedge T} e^{-\delta(s-t)}U_1(c(s))\mathrm{d}s + e^{-\delta(T-t)}U_2(W(T))1_{\{\iota>T\}}\right], \quad (2.3)$$

Where $E_{t,w}[\cdot] \triangleq E[\cdot|W(t)=w, \iota \geq t]$ $1_{\{\cdot\}}$ denotes the indicator function, and $U_1(\cdot)$ $U_2(\cdot)$ are two utility functions such that $U_i'(\cdot) > 0$ and $U_i''(\cdot) < 0$, $i = 1,2$. In terms of memoryless exponential distribution and conditioning the exponential distribution on $[t,T]$, it is not hard to show that the value function has the following equivalent form:

$$V(t,w) = \sup_{c(t),\beta(t)} E\left[\int_t^T e^{-(\delta+\lambda)(s-t)}U_1(c(s))\mathrm{d}s + e^{-(\delta+\lambda)(T-t)}U_2(W(T))|W(t)=w\right]. \quad (2.4)$$

We employ the technique of the principle of dynamic programming to solve the value function. Assume $V(t,w) \in \mathcal{C}^{1,1}$, then we can show that the value function satisfies the Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{c,\beta}\left\{\frac{\partial}{\partial t}V(t,w) - (\delta+\lambda+\kappa)V(t,w) + [rw - c + I(t) - (1+\theta)\nu\kappa\beta w]\frac{\partial}{\partial w}V(t,w)\right.$$
$$\left. + \kappa V(t,(1+(\beta-1)\nu)w) + U_1(c)\right\} = 0 \quad (2.5)$$

With the boundary condition $V(T,w) = U_2(w)$.

In order to obtain a closed-form solution for the value function, we also follow the usual assumption that the individual has a constant relative risk aversion utility preference. We let

$$U_1(x) = \frac{x^\gamma}{\gamma}, \quad (2.6)$$

be the wage earner's utility for consumptions and let

$$U_2(x) = \alpha\frac{x^\gamma}{\gamma}, \quad (2.7)$$

Be the wage earner's utility for terminal wealth, where $0 < \alpha < \infty$ is a relative weight coefficient. Here we require $0 < \gamma < 1$ to ensure the positive value of the utility function.

With the boundary conditions $V(T,w) = U_2(w)$, we speculate that the value function has the following form,

$$V(t,w) = \frac{a(t)}{\gamma}(w + b(t))^\gamma \quad (2.8)$$

Where $a(t)$ and $b(t)$ are deterministic functions to be determined. Once again, according to $V(T,w) = U_2(w)$, we have the boundary conditions

$$a(T) = \alpha \text{ and } b(T) = 0.$$

In addition, by direct calculations, we can obtain that

$$\frac{\partial}{\partial t}V(t,w) = \frac{a'(t)}{\gamma}(w+b(t))^\gamma + a(t)b'(t)(w+b(t))^{\gamma-1}, \quad (2.9)$$
$$\frac{\partial}{\partial w}V(t,w) = a(t)(w+b(t))^{\gamma-1}. \quad (2.10)$$

We now substitute (2.6), (2.8)-(2.10) into (2.5), having that

$$\sup_{c,\beta}\left\{\frac{a'(t)}{\gamma}(w+b(t))^{\gamma}+a(t)b'(t)(w+b(t))^{\gamma-1}-(\delta+\lambda+\kappa)\frac{a(t)}{\gamma}(w+b(t))^{\gamma}\right.$$
$$+[rw-c+I(t)-(1+\theta)\nu\kappa\beta w]a(t)(w+b(t))^{\gamma-1}$$
$$\left.+\kappa\frac{a(t)}{\gamma}[(1+(\beta-1)\nu)w+b(t)]^{\gamma}+\frac{c^{\gamma}}{\gamma}\right\}=0. \tag{2.11}$$

By the first-order condition, we obtain that the optimal consumption and insurance coverage should satisfy

$$a(t)(w+b(t))^{\gamma-1}=c^{*\gamma-1}, \tag{2.12}$$
$$(1+\theta)(w+b(t))^{\gamma-1}=[(1+(\beta-1)\nu)w+b(t)]^{\gamma-1} \tag{2.13}$$

which leads to the solution

$$c^{*}=[a(t)]^{\frac{1}{\gamma-1}}(w+b(t)), \tag{2.14}$$
$$\beta^{*}=1-\frac{w+b(t)}{\nu w}[1-(1+\theta)^{\frac{1}{\gamma-1}}]. \tag{2.15}$$

We substitute (2.14) and (2.15) into (2.11) and simplify it, thereby having

$$\left[\frac{a'(t)}{\gamma}-Aa(t)+(\frac{1}{\gamma}-1)a(t)^{\frac{\gamma}{\gamma-1}}\right](w+b(t))+[b'(t)-(r-(1+\theta)\kappa\nu)b(t)+I(t)]a(t)=0, \tag{2.16}$$

where

$$A=\frac{\delta+\lambda+\kappa}{\gamma}-r-(1+\theta)\kappa(1-\nu)+(1-\frac{1}{\gamma})\kappa(1+\theta)^{\frac{\gamma}{\gamma-1}}.$$

Since (2.16) holds for any w, we must have

$$\frac{a'(t)}{\gamma}-Aa(t)+(\frac{1}{\gamma}-1)a(t)^{\frac{\gamma}{\gamma-1}}=0, \tag{2.17}$$
$$b'(t)-[r-(1+\theta)\kappa\nu]b(t)+I(t)=0. \tag{2.18}$$

With boundary conditions $a(T)=\alpha$ and $b(T)=0$, we have

$$a(t)=\left[\alpha^{\frac{1}{1-\gamma}}+\frac{1-\gamma}{\gamma A}\left(1-\exp\left\{\frac{\gamma A}{1-\gamma}(t-T)\right\}\right)\right]^{1-\gamma}, \tag{2.19}$$
$$b(t)=\int_t^T\exp\{-(r-(1+\theta)\kappa\nu)(s-t)\}I(s)\mathrm{d}s. \tag{2.20}$$

Note that both $a(t)$ and $b(t)$ are positive.

By the conventional methods, we solve the HJB equation (2.5) and obtain the following theorem:

Theorem 2.1

For the problem (2.3), the optimal consumption and insurance policies are[1]

$$c_t^{*}=a(t)^{\frac{1}{\gamma-1}}(w+b(t)),$$
$$\beta_t^{*}=1-\frac{w+b(t)}{\nu w}[1-(1+\theta)^{\frac{1}{\gamma-1}}].$$

And the value function is positive such that

$$V(t,w)=\frac{a(t)}{\gamma}(w+b(t))^{\gamma},$$

Where $a(t)$ and $b(t)$ are given by (2.19) and (2.20), respectively.

[1] Obviously, it follows that $\beta_t^{*}<1$, but we can not guarantee that $\beta_t^{*}>0$ in general. For the special case that $I(s)=0$ and $\nu=1$, we have $0<\beta_t^{*}=(1+\theta)^{\frac{1}{\gamma-1}}<1$.

III. The case of ambiguous hazard rate

Due to limited knowledge about the risks, an individual can not be fully aware of the hazard rate about her/his future life, and all decisions are made based on subjective beliefs. Mathematically, the individual might be able to approximate her/his future life expectancy in the space $(\Omega, \mathcal{F})$. However, the individual can't exactly obtain the distribution of the death time. Thus the value λ in the previous section can be seen as a reference hazard rate for the individual, and the probability measure P can be seen as a reference probability measure (reference model). Intuitively, in order to allow the ambiguity of lifetime distribution, alternative hazard rates, as well as alternative probability measures, should be considered by the individual. In this case, the robust consumption policy and non-life insurance demand for the individual should be studied with consideration of alternative probability measures under the ambiguous case.

In order to choose suitable alternative probability measures, a Poisson process is given to act as a "selection tool" in this section. It is well known that the exponential random variable ι (death time) can be seen as the time to the first jump for a homogenous Poisson process with a const jump intensity λ. Assume that a Poisson process $N_1 = \{N_{1t}\}_{t\geq 0}$ with jump intensity λ is defined in the probability space $(\Omega, \mathcal{F}, P)$. Since the probability measure P as well as ι are ambiguous, the rate $N_1 = \{N_{1t}\}_{t\geq 0}$ is also ambiguous for the individual. It is obvious that if we obtain the alternative rates N_1 under alternative probability measures, the alternative hazard rates of the individual's lifetime follow. In addition, the reference model is the best description of the data information. Thus the considered alternative models should be similar to the reference model. By this motivation, we define the alternative models by a class of probability measures that are equivalent[2] to P:

$$\mathbb{Q} := \{Q : Q \sim P\}.$$

According to Girsanov's theorem, Q satisfies,

$$\frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = \Lambda(t), \tag{3.1}$$

where

$$\Lambda(t) = \exp\left\{\int_0^t \ln m(s)\mathrm{d}N_1(s) + \lambda\int_0^t (1 - m(s))\mathrm{d}s\right\}$$

Is a P-martingale with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, where $m(t)_{\{t\geq 0\}}$ is a $(\mathcal{F}_t)_{t\in[0,\infty)}$-progressively measurable positive process. By Girsanov's theorem, the process N_1 still is a Poisson process with jump intensity $\lambda m(t)$ under the alternative model Q, i.e., $\lambda^Q = \lambda m(t)$. Since the death time ι can be seen as the first jump of N_1, ι is an exponential distribution with hazard rate λ^Q under Q.

For the purpose of considering the alternative model Q, we here measure the discrepancy between each alternative model and the reference model by using the relative entropy. The relative entropy is a well-established approach in measuring the discrepancy between probability measures Q and P. For example, see Maenhout (2004). The relative entropy between Q and P is defined by

$$H_{[0,t]}(Q \parallel P) = E^Q_{[0,t]}\left[\ln\frac{\mathrm{d}Q}{\mathrm{d}P}\right] = E^Q_{[0,t]}\left[\int_0^t \lambda[m(s)\ln m(s) - m(s) + 1]\mathrm{d}s\right], \tag{3.2}$$

In which we define $\mathcal{Z}(s) := \lambda[m(s)\ln m(s) - m(s) + 1]$.

Now we are ready to construct a robust control problem for the individual as below. Although the reference probability measure P as well as the referenced hazard rate λ may be misspecification, they indeed provide the reference value. Thus, if we construct a problem with regard to Q instead of P, a penalty will incur. Intuitively, the more difference between Q and P is, the larger penalty will happen. Under this circumstance, the value function, still denoted by $V(t, w)$, is defined as

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[2] In a probability space, two measures P and Q are equivalent, denoted by $Q \sim P$, if they have the same null sets, i.e. $Q(A) = 0$ if and only if $P(A) = 0$ for $A \in \mathcal{F}$.

$$V(t,w)=\sup_{c(t),\beta(t)}\inf_{Q\in\mathbb{Q}}\Big\{E^Q_{t,w}\big[\int_t^{T\wedge\iota}e^{-\delta(s-t)}U_1(c(s))\mathrm{d}s+e^{-\delta(T-t)}U_2(W(T))1_{\{\iota>T\}}$$
$$+\int_t^{T\wedge\iota}e^{-\delta(s-t)}\xi\phi(V^Q(s,W(s))\mathcal{Z}(s))\mathrm{d}s\big]\Big\}, \tag{3.3}$$

where $E^Q_{t,w}[\cdot]$ is condition expectation under the probability measure Q, i.e.,

$$E^Q_{t,w}[\cdot]=E^Q[\cdot\mid W(0)=\omega,\iota>t],$$

$U_1(\cdot)$ And $U_2(\cdot)$ are individual's utility functions with the same definition as before, for an alternative model, Q the last item $\int_t^{T\wedge\iota}\xi\phi(V^Q(s,W(s))\mathcal{Z}(s))\mathrm{d}s$ denotes the penalty function, $\phi(\cdot)>0$ is a normalization factor that converts the penalty to the same order of magnitude as the order of $V^Q(t,w)$, and the constant $\xi>0$ denotes the degree of the individual's confidence in the reference model P. The larger the ξ is, the more confidence the individual will have on P.

According to the technique of dynamic programming principle, if $V(t,w)\in\mathcal{C}^{1,1}$, then we can give HJB equation that is satisfied by the value function (see Fleming and Soner (2006))

$$\sup_{c,\beta}\inf_{m}\Big\{\frac{\partial}{\partial t}V(t,w)-(\lambda m+\delta+\kappa)V(t,w)+[rw-c+I(t)-(1+\theta)\nu\kappa\beta w]\frac{\partial}{\partial w}V(t,w)$$
$$+\kappa V((1+(\beta-1)\nu)w,t)+U_1(c)+\lambda\xi(m\ln m-m+1)\phi(V(t,w))\Big\}=0, \tag{3.3}$$

with boundary conditions $V(T,w)=U_2(w)$.

Lemma 3.1 (Verification Theorem) If $\nu(t,w)\in\mathcal{C}^{1,1}$ is the solution to the HJB equation (3.3) with the boundary conditions $\nu(T,w)=U_2(w)$, then $\nu(t,w)=V(t,w)$.

Proof. It is similar to the Lemma 3.2. of Liu and Zhou (2021). So we omit it here.

In the following, we provide closed-form solutions to the optimal problem (3.2). For analytical convenience, we take a special case in (3.2) and (3.3) that

$$\phi(x)=x. \tag{3.4}$$

As mentioned by Maenhout (2004), the reason for taking the special case is to ensure the homotheticity or scale invariance of the value function and, therefore, to ensure that the optimal decision problem (3.2) has a natural economic justification. Because the relative entropy $H_{[0,t]}(Q\parallel P)$ is actually unitless, a suitable form $\phi(\cdot)$ can convert the penalty to the same order of magnitude as the order of the value function.

Substituting (3.4) into (3.3), we have the HJB equation

$$\sup_{c,\beta}\inf_{m}\Big\{\frac{\partial}{\partial t}V(t,w)-(\lambda m+\delta+\kappa)V(t,w)+[rw-c+I(t)-(1+\theta)\nu\kappa\beta w]\frac{\partial}{\partial w}V(t,w)$$
$$+\kappa V((1+(\beta-1)\nu)w,t)+U_1(c)+\lambda\xi(m\ln m-m+1)V(t,w)\Big\}=0. \tag{3.5}$$

First of all, we assume that the value function $V(t,w)$ is positive, which can be verified later. In accordance with the first-order conditions, we can obtain that m^* has the following form

$$m^*=\exp(\frac{1}{\xi}). \tag{3.6}$$

Inserting (3.6) in (3.5), which yields

$$\sup_{c,\beta}\left\{\frac{\partial}{\partial t}V(t,w)+[rw-c+I(t)-(1+\theta)\nu\kappa\beta w]\frac{\partial}{\partial w}V(t,w)+U_1(c)\right.$$
$$\left.+\kappa V((1+(\beta-1)\nu)w,t)+[\lambda\xi(1-\exp(\frac{1}{\xi}))-\delta-\kappa]V(t,w)\right\}=0. \quad (3.7)$$

In order to solve (3.7), we take a trial solution

$$V(t,w)=\frac{A(t)}{\gamma}(w+B(t))^{\gamma}, \quad (3.8)$$

where $A(t)$ and $B(t)$ are deterministic functions to be determined. Since $V(T,w)=U_2(w)$we have $A(T)=\alpha$ and $B(T)=0$. Therefore, we have

$$\frac{\partial}{\partial t}V(t,w)=\frac{A'(t)}{\gamma}(w+B(t))^{\gamma}+A(t)B'(t)(w+B(t))^{\gamma-1}, \quad (3.9)$$
$$\frac{\partial}{\partial w}V(t,w)=A(t)(w+B(t))^{\gamma-1}. \quad (3.10)$$

We now substitute (2.6), (3.8)-(3.10) into (3.7), obtaining that

$$\sup_{c,\beta}\{\frac{A'(t)}{\gamma}(w+B(t))^{\gamma}+A(t)B'(t)(w+B(t))^{\gamma-1}+[rw-c+I(t)-(1+\theta)\nu\kappa\beta w]A(t)(w+B(t))^{\gamma-1}$$
$$+[\lambda\xi(1-\exp(\frac{1}{\xi}))-\delta-\kappa]\frac{A(t)}{\gamma}(w+B(t))^{\gamma}+\kappa\frac{A(t)}{\gamma}[(1+(\beta-1)\nu)w+B(t)]^{\gamma}+\frac{c^{\gamma}}{\gamma}\}=0. \quad (3.11)$$

By the first condition again, we have

$$c^*=[A(t)]^{\frac{1}{\gamma-1}}(w+B(t)), \quad (3.12)$$
$$\beta^*=1-\frac{w+B(t)}{\nu w}[1-(1+\theta)^{\frac{1}{\gamma-1}}]. \quad (3.13)$$

We substitute (3.12) and (3.13) into (3.11) and simplify it, thereby having

$$[\frac{A'(t)}{\gamma}-\mathcal{A}A(t)+(\frac{1}{\gamma}-1)A(t)^{\frac{\gamma}{\gamma-1}}](w+B(t))+[B'(t)-(r-(1+\theta)\kappa\nu)B(t)+I(t)]A(t)=0, \quad (3.14)$$

where[3]

$$\mathcal{A}=\frac{\lambda\xi(\exp(\frac{1}{\xi})-1)+\delta+\kappa}{\gamma}-(1+\theta)\kappa(1-\nu)-r+(1-\frac{1}{\gamma})\kappa(1+\theta)^{\frac{\gamma}{\gamma-1}}.$$

Since (3.14) holds for all w, we have

$$\frac{A'(t)}{\gamma}-\mathcal{A}A(t)+(\frac{1}{\gamma}-1)A(t)^{\frac{\gamma}{\gamma-1}}=0, \quad (3.15)$$
$$B'(t)-(r-(1+\theta)\kappa\nu)B(t)+I(t)=0. \quad (3.16)$$

With $A(T)=\alpha$ and $B(T)=0$, we have

$$A(t)=\left[\alpha^{\frac{1}{1-\gamma}}+\frac{1-\gamma}{\gamma\mathcal{A}}\left(1-\exp\left\{\frac{\gamma\mathcal{A}}{1-\gamma}(t-T)\right\}\right)\right]^{1-\gamma}, \quad (3.17)$$
$$B(t)=\int_t^T\exp\{-(r-(1+\theta)\kappa\nu)(s-t)\}I(s)\mathrm{d}s=b(t). \quad (3.18)$$

Note that both $A(t)$ and $B(t)$ are positive.

Thus far, we have obtained the optimal consumption policy in the case of ambiguous hazard rate, which can be concluded as the following theorem:

Theorem 3.1. For problem (3.2), the optimal consumption policy and non-life insurance demand[4] are

[3] Letting $\xi\to+\infty$, it follows that $\mathcal{A}\to A$, which reduces to the case without lifetime ambiguity in Section 2.

[4] We can see from the result that the optimal non - life insurance demand β^* does not be affected by the individual's lifetime

$$c^* = [A(t)]^{\frac{1}{\gamma-1}}(w + b(t)),$$
$$\beta^* = 1 - \frac{w + b(t)}{\nu w}[1 - (1+\theta)^{\frac{1}{\gamma-1}}].$$

And the value function is positive such that

$$V(t,w) = \frac{A(t)}{\gamma}(w + b(t))^\gamma$$

Where $A(t)$ and $b(t)$ are given by (3.17) and (3.18), respectively.

IV. Numerical calculations

In this section, we present several numerical examples to examine the economic implications of our explicit solutions. Unless otherwise stated, we take the following values for the parameters: $w = 400$, $T = 60$, $t = 40$, $I(t) = 40$, $r = 0.04$, $\lambda = 0.05$, $\alpha = 1$, $\gamma = 0.2$, $\kappa = 0.1$, $\nu = 0.5$.

First of all, let's see the impacts of parameters on the non-life insurance demand, which is displayed in Figure 1. Recall the optimal insurance demand

$$\beta^* = 1 - \frac{w + b(t)}{\nu w}[1 - (1+\theta)^{\frac{1}{\gamma-1}}].$$

As mentioned in the last section, we firstly know that the individual's lifetime ambiguity does not affect the insurance demand amount. Because $b(t)$ it is positive and it decreases with respect to t, the optimal insurance demand is increasing with the age of the individual. In addition, we know $b(t)$ she is increasing with respective parameters κ, so the optimal insurance demand is decreasing with respect to the frequency of wealth loss. In other words, the more frequently the individual loses wealth, the less insurance the individual should buy for each loss. And $b(t)$ is decreasing with respect to the risk-free interest rate, so the optimal insurance demand is increasing with respect to the risk-free interest rate r. At last, for wealth w, the more wealth the individual holds, the more insurance is needed. As for other parameters, we can't directly see their impacts on the insurance demand from the expression. Figure 1 shows the impacts of the maximum loss proportion ν, the safe loading of the insurance premium θ , and the frequency of wealth loss κ on optimal insurance demand policy. We can see from Figure 1 that the optimal insurance demand proportion β is between 0 and 1 for reasonable parameters. On the left graph of Figure 1, we can see that ν it has a positive effect on the insurance policy. It can be summarized that the individual tends to buy more insurance when he/she faces a larger loss amount. On the middle graph of Figure 1, we can see that the optimal insurance policy β^* decreases with θ. This is because that the larger the θ is, the more expensive the price of the insurance is. The right graph shows that when the loss frequency is higher (larger κ), the individual will buy less insurance. This is because that higher loss frequency causes a larger premium. In order to limit premium payments, the individual has to reduce the insurance coverage to hedge the higher loss frequency.

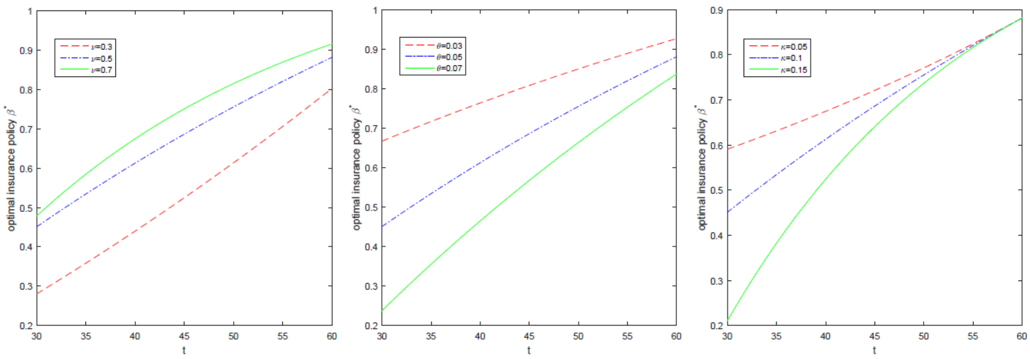

Figure 1: Impacts of ν, θ and κ on optimal insurance policy

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ambiguity due to the independence of them.

Secondly, let's see the impacts of parameters on robust consumption policy. In Figure 2, we can see that the optimal consumption policy decreases firstly and then increases as the time t increases from 30 to 60. It can be concluded that, for the individual, in the youth, she/he consumes less and saves more, but the consumption is increasing between middle age and retirement. This consumption model is completely consistent with people's consumption behavior. From the first and the second graphs of Figure 2, we can see that the optimal consumption policy has an increasing tendency with respect to the maximum loss proportion ν and the frequency of wealth loss κ. This result means that the individual will consume more if she/he faces either a larger risk for each loss or more frequency the loss occurs. The third graph in Figure 2 reveals that the impact of safety loading of premium θ is not significant. In the youth, we can see that the higher the safety loading, the more the individual consumes. The possible reason is that higher insurance price reduces the motivation that the individual buys insurance, so she/he would like to consume a little bit more. The last graph of Figure 2 shows that the optimal consumption increases with the increase of λ. It means that the individual is likely to consume more with the decreasing expectation of a lifetime $1/\lambda$.

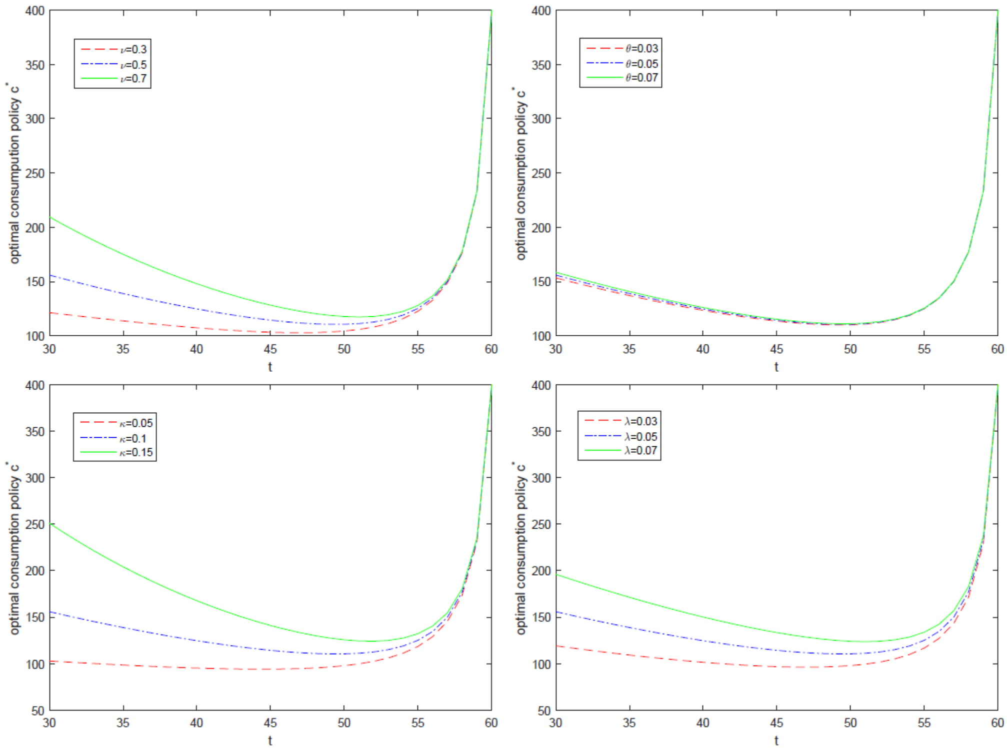

Figure 2: Impacts of ν, θ, κ and λ on optimal consumption policy

At last, Figure 3 displays the impact of ambiguity aversion on the optimal consumption policy. As we can see, the optimal consumption policy is decreasing with respect to the coefficient ξ. In other words, the individual would like to spend more money on consumption if she/he is more uncertain about the reference model, that is, the more pessimistic about the future lifetime. The conclusion is completely consistent with our intuition.

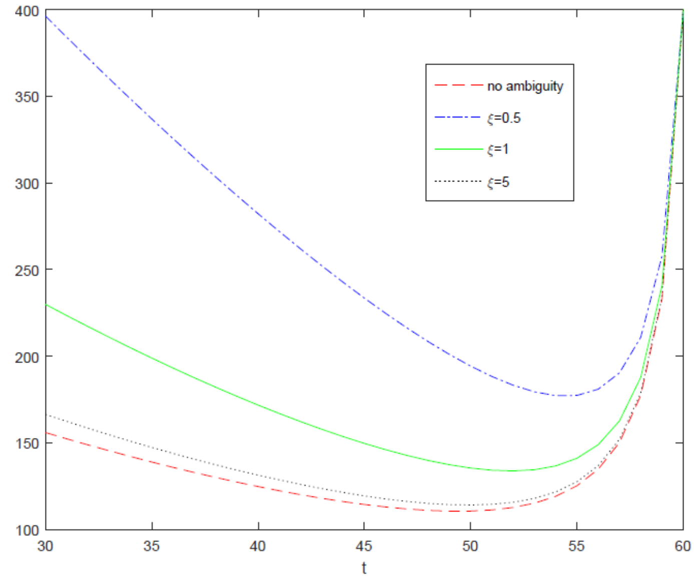

Figure 3: Impacts of ξ on optimal consumption policy

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