Completion Time of Special Kind of GERT-Type Networks with Fuzzy Times for Activities

S.Tousheh Asl[#1], S.S.Hashemin [*2]

[#]Department of Industrial Engineering, Ardabil Branch, Islamic Azad University, Ardabil, Iran
[*]Department of Industrial Engineering, Ardabil Branch, Islamic Azad University, Ardabil, Iran

Abstract — *Due to uncertainty in realization of some of the project activities, GERT-type networks are used. Also in these networks, duration of activities is non-deterministic. In this research, it is assumed that the probability of activity realization is known. Also, it is supposed that the activity durations can be shown as a positive trapezoidal fuzzy number. Studied networks don't have any loops. They have one source node and they can have more than one target node. Based on simulation, a new algorithm is developed for acquisition of all sub networks of the project network. It is obvious that the completion time of project is a fuzzy random variable. So, the probability function and cumulative distribution function of completion time of the project are defined. These functions are obtained for each end node of the project. Therefore, mean and variance of completion times can be computed. Finally, an example has been solved using the new algorithm.*

Keywords — *GERT-type networks, Trapezoidal fuzzy number, Fuzzy random variable, Project completion time.*

I. Introduction

In many of the real-world projects, not only the duration of activities but also their realization time is also non-deterministic. Usually, the duration of activities is estimated using experts' experience. Expressions like roughly, almost, more or less indicate uncertainty in the estimation of parameters. Fuzziness and randomness are two basic types of uncertainty. In many cases, fuzziness and randomness simultaneously appear in a system. In other words, a random variable can have fuzzy values. Studying the completion time of the network is important, because the completion of the project on time can affect the cost. Some new analytical methods for determining the completion time of GERT-type networks have been proposed by [1], [2]. The duration of activities is assumed a random variable in [3], [4]. Gavareshki [5] proposes a new applicable technique for research project scheduling. In this project, nodes are fuzzy and output activities from nodes of network belong to a fuzzy set. This method computes the network completion time as a fuzzy number. Liu et al [6] believe that the traditional GERT networks cannot reflect the characteristics of real-world network problems and uses the triangular fuzzy numbers to formulate the fuzzy GERT model.

Fuzzy completion time for alternative stochastic networks has been studied. The intended network contains only nodes with exclusive-or receiver and exclusive-or emitter. Completion time of network is a fuzzy valued random variable. The expected completion time of network is computed as a trapezoidal fuzzy number [7].

If the moment-generating function of random variable of activity time cannot be defined, then the topology equation or maison method cannot be utilized. If we use the topology equation or simplification method, only the moments of network completion time can be calculated. Calculating the high order moments is difficult. So, the first and second moments are usually calculated. But the probability distribution function of network completion time is not computable. If activity duration has special distribution such as Cauchy, then we will not be able to use the topology equation or maison method because the moment-generating function for this distribution is not defined. As a result of this, fuzzy numbers are considered to describe the activity duration. In this paper, it is supposed that the activity duration is shown with a positive trapezoidal fuzzy number.

The paper has the following structure: Section Ⅱ introduces the definitions, operations and assumptions. Notations are introduced in section Ⅲ. Section Ⅳ introduces the proposed algorithm and describes its steps. Section Ⅴ describes the stop criteria of algorithm. Section Ⅵ explains the ranking method. Section Ⅶ illustrates the proposed algorithm using an example. Finally section Ⅷ is devoted to conclusion and recommendations for future studies.

II. Definitions, Operations and Assumptions

A. Definitions

In this section, some basic notions of fuzzy theory that have been defined in [8] are introduced.

Definition 1: Let R be the set of real numbers. A fuzzy set $\tilde{A}$ is a set of ordered pairs $\{(x, \mu_{\tilde{A}}(x)) | x \epsilon X\}$, where $\mu_{\tilde{A}}(x)$: R → [0,1] and is called membership function of the fuzzy set.

Definition 2: A convex fuzzy set, $\tilde{A}$, is a fuzzy set in which:

$\forall x, y \in R$, $\forall \lambda \in [0,1]$,

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq min[\mu_{\tilde{A}}(x), \quad \mu_{\tilde{A}}(y)]$$

Definition 3: A fuzzy set $\tilde{A}$ is called positive if its membership function is such that:

$\mu_{\tilde{A}}(x) = 0, \forall x \leq 0$.

Definition 4: Trapezoidal fuzzy number (TFN) is a convex fuzzy set, which is defined as:

$\tilde{A} = (x, \mu(x))$ Where:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & b < x \leq c \\ \frac{x-d}{c-d} & c < x \leq d \\ 0 & x \geq d \end{cases}$$

For convenience, TFN represented by four real parameters a,b,c,d $(a \leq b \leq c \leq d)$ will be denoted by a trapezoid (a, b, c, d) (Fig. 1).

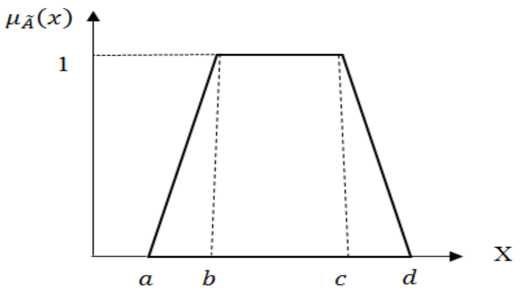

Fig 1: Trapezoidal fuzzy number (TFN)

Definition 5: A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ is called positive TFN if:

$0 \leq a \leq b \leq c \leq d$.

B. Operations on TFNs

Some of operations can be performed on TFNs. Fuzzy addition; fuzzy multiplication and fuzzy scalar multiplication have been defined in [9].

Fuzzy addition:

Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2)$ be any two TFNs, then:

$\tilde{A} \oplus \tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$

Fuzzy multiplication:

$\tilde{A} \otimes \tilde{B} = (a_1 \times a_2, b_1 \times b_2, c_1 \times c_2, d_1 \times d_2)$

Fuzzy scalar multiplication: Let $\tilde{A} = (a_1, b_1, c_1, d_1)$ be any TFN and q be a real number, then:

$q \odot \tilde{A} = (qa_1, qb_1, qc_1, qd_1)$

Where $\oplus$= fuzzy addition; $\otimes$= fuzzy multiplication; $\odot$= fuzzy scalar multiplication.

C. Assumptions

- The network has a single source node and it can have one or more sink node.
- The network does not contain any loops.
- The probability of network activity realization is known.
- Duration of the network activities is a positive TFN.
- There are several types of nodes in the network.

III. Notations

The following notations have been used in this paper:

$\tilde{D}_K$ Duration time of k-th activity (It is a positive TFN),

$\bar{P}_K$ Accomplishment probability of k-th activity, given that start node of this activity has been realized,

M Number of sink nodes,

n_i Number of paths which start from source node and terminate in i-th sink node,

$\tilde{T}_{ij}$ The completion time of sub network j-th which terminates in i-th sink node,

A_{ij} Sub network j-th which terminates in i-th sink node,

P_{ij} Realization probability of j-th sub network which terminates in i-th sink node,

$\tilde{T}$ Completion time of the network,

P_i Realization probability of i-th sink node,

$P(\tilde{T} = \tilde{T}_{ij})$ Fuzzy random variable probability function of the network completion time,

$P((\tilde{T} = \tilde{T}_{ij} | i))$ Fuzzy random variable probability function of the network completion time, given that, the node i is sink node,

$F_{\tilde{T}}(t)$ Fuzzy random variable cumulative distribution function of the network completion time,

$F_{\tilde{T}}(t|i)$ Fuzzy random variable cumulative distribution function of the network completion time, given that, the node i is sink node.

IV. Proposed Algorithm

Step 1: $\tilde{T}_S = (0,0,0,0)$

Step 2: If source is the start node of one activity then suppose that the no. of the activity is k and execute the activity.

$$\tilde{T}_E = \tilde{T}_S \oplus \tilde{D}_K, \quad (1)$$

Then the end node of the activity is realized. If this node is one of the end nodes of the network, stop. Otherwise, end node of the activity is start node of other activities. In this case, repeat step 2.

If the emitter part of start node is PROBABILISTIC, and the node is the start node of several activities then by generating a random number from uniform distribution between 0, 1, the realized activity is identified. If the number of the realized activity is k then execute the activity. So, we have:

$$\tilde{T}_E = \tilde{T}_S \oplus \tilde{D}_K, \quad (2)$$

Then the end node of the activity is realized. If this node is one of the end nodes of the network, stop. Otherwise, end node of the activity is start node of other activities. In this case, repeat step 2.

If the output of start node is AND, then all output activities will be realized. The realized activities are divided into Z subsets according to the end nodes. In other words, activities of subset $L_z, z = 1, \dots, Z$ have common end node. So, we will have three cases.

1) If the input of this node is EXCLOSIVE-OR, then L_z have one member. We will have:

$$\tilde{T}_E = \tilde{T}_{S_i} \oplus \tilde{D}_K, \quad (3)$$

The EXCLUSIVE-OR node will be realized. If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

2) If input of end node related to subset L_z is INCLUSIVE-OR, then we will have:

$$\tilde{T}_E = \tilde{T}_{S_i} \oplus min_{i \in L_z}\{\tilde{D}_i\}, \quad (4)$$

If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

3) If input of end node related to subset L_z is AND, then we will have:

$$\tilde{T}_E = \tilde{T}_{S_i} \oplus \max_{i \in L_z}\{\tilde{D}_i\}, \quad (5)$$

If this node is one of the end nodes of the network, stop. Otherwise, go to step 3.

Step 3: For newly realized nodes, define the list of all activities which are started from above mentioned nodes. Again, the sum of the realized activities is divided into Z subsets according to the end nodes.

Calculations for each of the subsets depending on input part of end node will be one of the following:

$$\tilde{T}_E = \tilde{T}_{S_i} \oplus \tilde{D}_i \quad , \quad (6)$$

$$\tilde{T}_E = \max_i\{\tilde{T}_{S_i} \oplus \tilde{D}_i\}, \quad (7)$$

$$\tilde{T}_E = \min_i\{\tilde{T}_{S_i} \oplus \tilde{D}_i\}, \quad (8)$$

$\tilde{T}_{S_i}$ Start time of activity or (activities i)

If one of the end nodes of the network is realized by the realization of the activities of one of the subsets, stop and go to step 4. Otherwise repeat this step. At each step of the algorithm, we record the realized activities. Finally, using the list of the realized activities, the realized sub network can be identified and can be drawn.

Step 4: Obtained $\tilde{T}_E$ for end node i is the completion time of sub network j which terminates in i-th sink node. Occurrence probability of each node is computed and these operations are repeated while the occurrence node is one of the end nodes of the network. Finally, occurrence probability of sub network j which terminates in i-th sink node is obtained.

V. Stop Criteria of Algorithm

By repeating the simulation, other sub networks will be obtained. We should check the sum of probabilities of sub networks occurrence in each simulation run. If this summation is equal with one, then stop. Otherwise, repeat the simulation.

The probability function, average, cumulative distribution function and variance will be calculated respectively as follows:

$$P(\tilde{T} = \tilde{T}_{ij}) = P_{ij} \ , \ \forall \, i, j \ , \quad (9)$$

$$E(\tilde{T}) = \sum_{i=1}^{M} \oplus \sum_{j=1}^{n_i} \oplus \ (P_{ij} \odot \tilde{T}_{ij}), \quad (10)$$

$$F_{\tilde{T}}(t) = \sum_{\tilde{T}_{ij} \le t} P(\tilde{T} = \tilde{T}_{ij}), \quad (11)$$

$$\sigma^2 = E(\tilde{T} \otimes \tilde{T}) \ominus [E(\tilde{T}) \otimes E(\tilde{T})], \quad (12)$$

Where $\ominus$ shows the fuzzy subtraction.

We can also calculate $P((\tilde{T} = \tilde{T}_{ij}|i))$ and $F_{\tilde{T}}(t|i)$.

It is evident that:

$$E(\tilde{T}) \otimes E(\tilde{T}) = (a_1, b_1, c_1, d_1), \quad (13)$$

$$E(\tilde{T} \otimes \tilde{T}) = \sum_{i=1}^{M} \oplus \sum_{j=1}^{n_i} \oplus \ \left(P_{ij} \odot (\tilde{T}_{ij} \otimes \tilde{T}_{ij})\right), \quad (14)$$

$$E(\tilde{T} \otimes \tilde{T}) = (a_2, b_2, c_2, d_2).$$

A new subtraction operator for positive trapezoidal number was defined such that the subtraction of a positive trapezoidal number is positive [10]. This operator is defined as follows:

$$d = \max(0, (d_2 - d_1)),$$

$$c = \max\left(0, min(d, (c_2 - c_1))\right), \quad (15)$$

$$b = \max\left(0, min(c, (b_2 - b_1))\right),$$

$$a = \max\left(0, min(b, (a_2 - a_1))\right),$$

$$\sigma^2 = (a, b, c, d).$$

Also we can calculate the conditional probability function and conditional cumulative distribution function for each network end node.

$$P((\tilde{T} = \tilde{T}_{ij}|i)) = \frac{P(\tilde{T}=\tilde{T}_{ij})}{P_i}, \quad (16)$$

$$F_{\tilde{T}}(t|i) = P(\tilde{T} \leq t|i). \quad (17)$$

VI. Ranking Fuzzy Numbers by Distance Method

Several different methods have been proposed for ranking fuzzy numbers. Cheng [11] proposed the distance method for ranking fuzzy numbers,

$$R(u) = \sqrt{\bar{x}^2 + \bar{y}^2},$$

Where

$$\bar{x} = \frac{\int_a^b x\, u_L dx + \int_b^c x dx + \int_c^d x\, u_R dx}{\int_a^b u_L dx + \int_b^c dx + \int_c^d u_R dx}, \quad \bar{y} = \frac{\int_0^1 r\, \underline{u} dr + \int_0^1 r\, \overline{u} dr}{\int_0^1 \underline{u} dr + \int_0^1 \overline{u} dr},$$

u_L, u_R are the left and right membership function of fuzzy number u, and $(\underline{u}, \overline{u})$ is the parametric form (see Definitions 1 and 2). The resulting scalar value $R(u)$ is used to rank the fuzzy numbers;

if $R(u_i) < R(u_j)$, then $u_i < u_j$. If $R(u_i) > R(u_j)$, then $u_i > u_j$; if $R(u_i) = R(u_j)$, then $u_i \sim u_j$.

Definition 1: A fuzzy number is a fuzzy set like u: R→ $I = [0,1]$which satisfies:

1) u is upper semi-continuous,

2) $u(x) = 0$ If x does not belong to$[a, d]$,

3) a, b,c,d are real numbers such that $a \leq b \leq c \leq d$ and

(a) $u(x)$ is monotonic increasing on $[a, b]$,

(b) $u(x)$ is monotonic decreasing on $[c, d]$,

(c) $u(x) = 1$, if $b \leq x \leq$c.

The membership function of u can be expressed as:

$$u(x) = \begin{cases} u_L(x) & a \leq x \leq b, \\ 1 & b \leq x \leq c, \\ u_R(x) & c \leq x \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2: A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfies the following requirements:

1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3) $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

The trapezoidal fuzzy number $u = (x_0, y_0, \sigma, \beta)$, with two defuzzifier x_0, y_0, and left fuzziness σ > 0 and right fuzziness β > 0 is a fuzzy set where the membership function is as below:

$$u(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0, \\ 1 & x \in [x_0, y_0], \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta, \\ 0 & \text{otherwise,} \end{cases}$$

And its parametric form is $\underline{u}(r) = x_0 - \sigma + \sigma r$, $\overline{u}(r) = y_0 + \beta - \beta r$.

Steps 2 and 3 of the proposed algorithm will be conducted using the ranking method.

VII. Example

The GERT network for a project has been shown in Fig. 2. Activity durations are positive trapezoidal numbers as shown in Table I.

TABLE I: $\tilde{D}_i$of the example

Activity	Duration ($\tilde{D}_i$)
1	(3,4,5,6)
2	(2,3,4,5)
3	(2,4,5,6)
4	(3,4,5,7)
5	(2,3,5,8)
6	(2,4,7,8)
7	(1,3,4,7)
8	(4,5,7,8)
9	(3,7,8,9)
10	(2.5,3,4,4.5)
11	(2,3,4,5)
12	(1.5,2.5,4.5,5)
13	(2,3,5,6)
14	(1.5,2,3,3.5)
15	(2,3,5,8)
16	(4,5,6,7)
17	(3,5,6,7)
18	(3,5,7,9)
19	(3.5,5,6,8)

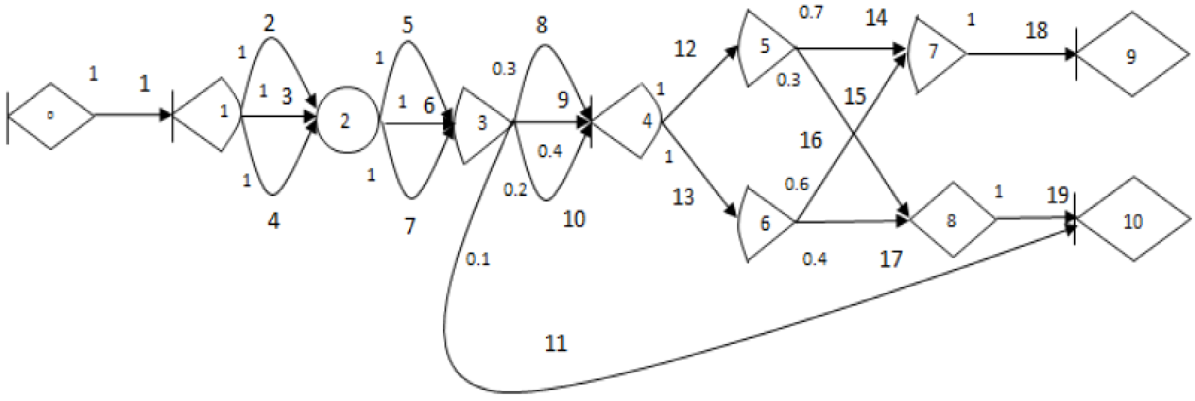

Fig 2: GERT network of the example

Using the proposed algorithm in previous sections, we have:

Step 1: $\tilde{T}_S = (0,0,0,0)$

Step 2: The realization time of node 1 using formula (1) is computed. Then we have:

$\tilde{T}_E = \tilde{T}_S \oplus \tilde{D}_1 = (0,0,0,0) \oplus (3,4,5,6) = (3,4,5,6),$

Because this node is not one of the end nodes of the network, repeat step 2.
Realized activity: 1,

Repeat step 2: The realization time of node 2 using formula (5) is computed. Then we have:
$L_1 = \{2,3,4\}$

$\tilde{T}_E = \tilde{T}_{S_i} \oplus \max_{i \in L_1}\{\tilde{D}_2, \tilde{D}_3, \tilde{D}_4\} = (3,4,5,6) \oplus (3,4,5,7)$
$= (6,8,10,13)$

The maximum of fuzzy numbers is defined by using the ranking method that was described in VI.
Because this node is not one of the end nodes of the network, go to step 3.

Realized activity: 2, 3, 4

Step 3: The realization time of node 3 using formula (7) is computed. Then we have:

$L_1 = \{5,6,7\}$

$$\tilde{T}_E = \max_{i \in L_1}\{\tilde{T}_{S_5} \oplus \tilde{D}_5, \tilde{T}_{S_6} \oplus \tilde{D}_6, \tilde{T}_{S_7} \oplus \tilde{D}_7\}$$
$$= (6,8,10,13) \oplus (2,4,7,8)$$
$$= (8,12,17,21)$$

Because this node is not one of the end nodes of the network, repeat step 3.

Realized activity: 5, 6, 7

Repeat step 3: The realization time of node 4 using formula (6) is computed. Then we have:

$L_1 = \{9\}$

$$\tilde{T}_E = \tilde{T}_{S_9} \oplus \tilde{D}_9 = (8,12,17,21) \oplus (3,7,8,9)$$
$$= (11,19,25,30)$$

Because this node is not one of the end nodes of the network, repeat step 3.
Realized activity: 9

Repeat step 3: The realization time of nodes 5, 6 using formula (7) are computed. Then we have:

$L_1 = \{12\}$
$L_2 = \{13\}$
$\tilde{T}_E = \max_{i \in L_1}\{\tilde{T}_{S_{12}} \oplus \tilde{D}_{12}\}$
$= max\{(11,19,25,30) \oplus (1.5,2.5,4.5,5)\}$
$= (12.5,21.5,29.5,35)$
$\tilde{T}_E = \max_{i \in L_2}\{\tilde{T}_{S_{13}} \oplus \tilde{D}_{13}\}$
$= max\{(11,19,25,30) \oplus (2,3,5,6)\} = (13,22,30,36)$

Since, nodes 5, 6 are not the end nodes of network, so these nodes are assumed to be start nodes in the next steps.
Realized activity: 12, 13

Repeat step 3: The realization time of node 8 using formula (8) is computed. Then we have:
$L_1 = \{17\}$

$$\tilde{T}_E = \min_{i \in L_1}\{\tilde{T}_{S_{17}} \oplus \tilde{D}_{17}\} = (13,22,30,36) \oplus (3,5,6,7)$$
$$= (16,27,36,43)$$

Because this node is not one of the end nodes of the network, repeat step 3.
Realized activity: 14, 17

Repeat step 3: The realization time of node 10 using formula (6) is computed. Then we have:
$L_1 = \{19\}$
$\tilde{T}_E = \tilde{T}_{S_{19}} \oplus \tilde{D}_{19} = (16,27,36,43) \oplus (3.5,5,6,8)$
$= (19.5,32,42,51)$

Because, this node is one of the end nodes of the network, stop and go to step 4.

Realized activity: 19

Step 4:

As a result: $\tilde{T}_E = \tilde{T}_{10,j}$. The network has two end nodes, so: M = 2, i = 1, 2 and $j = 1, \dots, 13$.

In Fig. 2, assume that node 9 is called end node 1 and node 10 is called end node 2. We have:

$\tilde{T}_{2,j} = (19.5,32,42,51)$

Continuing the simulation, by using the list of realized activities sub network can be defined. Fig. 3 shows the above mentioned sub network.

We can also calculate the realized probability of the sub network, so we have:

$P_{2,j}$ = 0.4×0.7×0.4= 0.112

In the following, based on the proposed algorithm, a computer program is written in the MATLAB software environment. The output of this program shows the list of the realized activities in tabular form. The realized activities using MATLAB software have been shown in Table Ⅱ. In Table Ⅱ, 1 shows the realized activity and 0 shows the unrealized activity. By recognizing the realized activities, required calculations are done. The Table Ⅲ shows the value of $\tilde{T}_{i,j}$ corresponds to $A_{i,j}$. P_{ij} Shows the occurrence probability of realized sub network that it is shown with $A_{i,j}$.

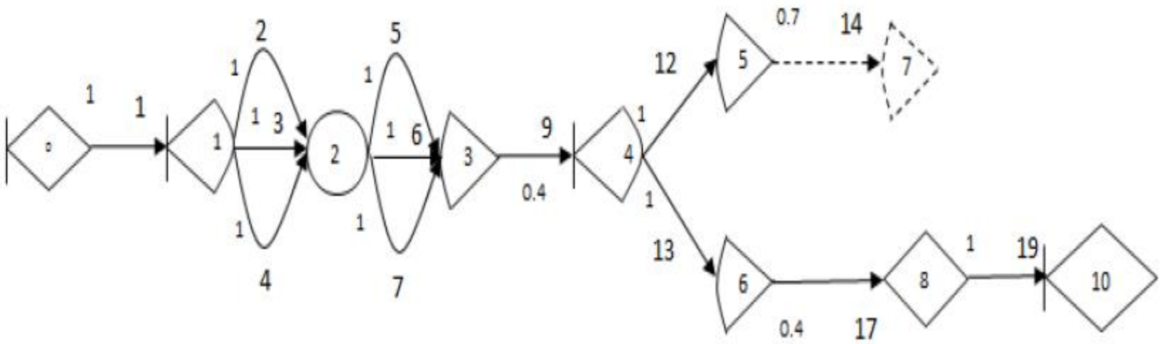

Fig 3: One of the identified sub networks

TABLE Ⅱ: The output of the MATLAB program for example

Activity / Sub network	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
A_{12}	1	1	1	1	1	1	1	0	1	0	0	1	1	1	0	1	0	1	0
A_{11}	1	1	1	1	1	1	1	1	0	0	0	1	1	1	0	1	0	1	0
$A_{2,12}$	1	1	1	1	1	1	1	0	0	1	0	1	1	0	1	1	0	0	1
A_{13}	1	1	1	1	1	1	1	0	0	1	0	1	1	1	0	1	0	1	0
A_{29}	1	1	1	1	1	1	1	0	1	0	0	1	1	1	0	0	1	0	1
$A_{2,10}$	1	1	1	1	1	1	1	0	1	0	0	1	1	0	1	1	0	0	1
A_{27}	1	1	1	1	1	1	1	1	0	0	0	1	1	1	0	0	1	0	1
A_{24}	1	1	1	1	1	1	1	1	0	0	0	1	1	0	1	0	1	0	1
$A_{2,11}$	1	1	1	1	1	1	1	0	0	1	0	1	1	1	0	0	1	0	1
A_{26}	1	1	1	1	1	1	1	0	0	1	0	1	1	0	1	0	1	0	1
$A_{2,13}$	1	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0
A_{25}	1	1	1	1	1	1	1	0	1	0	0	1	1	0	1	0	1	0	1
A_{28}	1	1	1	1	1	1	1	1	0	0	0	1	1	0	1	1	0	0	1

TABLE Ⅲ: The value of $\tilde{T}_{ij}$ related to A_{ij} and P_{ij}

A_{ij}	$\tilde{T}_{ij}$	P_{ij}
A_{12}	(20,32,43,52)	0.168
A_{11}	(21,30,42,51)	0.126
$A_{2,12}$	(17.5,25.5,36.5,46.5)	0.036
A_{13}	(19.5,28,39,47.5)	0.084
A_{29}	(19.5,32,42,51)	0.112
$A_{2,10}$	(18,29.5,40.5,51)	0.072
A_{27}	(20.5,30,41,50)	0.084
A_{24}	(19,27.5,39.5,50)	0.036
$A_{2,11}$	(19,28,38,46.5)	0.056
A_{26}	(17.5,25.5,36.5,46.5)	0.024
$A_{2,13}$	(10,15,21,26)	0.1
A_{25}	(18,29.5,40.5,51)	0.048
A_{28}	(19,27.5,39.5,50)	0.054

Using Table Ⅲ, the realization probability of the end node 1 and end node 2 can be calculated. P_1 And P_2 will be as follows:

$P_1 = P_{11} + P_{12} + P_{13} = 0.378$

$P_2 = P_{24} + P_{25} + P_{26} + P_{27} + P_{28} + P_{29} + P_{2,10} + P_{2,11} + P_{2,12} + P_{2,13} = 0.622$

The probability function of the network completion time is calculated using formula (9). Table IV shows the network probability function.

TABLE IV: Probability function of the network

$P(\tilde{T} = \tilde{T}_{ij})$	$\tilde{T}_{ij}$
0.1	(10,15,21,26)
0.06	(17.5,25.5,36.5,46.5)
0.056	(19,28,38,46.5)
0.084	(19.5,28,39,47.5)
0.09	(19,27.5,39.5,50)
0.12	(18,29.5,40.5,51)
0.084	(20.5,30,41,50)
0.112	(19.5,32,42,51)
0.126	(21,30,42,51)
0.168	(20,32,43,52)

The expected value of network completion time is calculated using Table IV and formula (10).

$E(\tilde{T}) = 0.1 \odot (10, 15, 21\ 26)$
$\oplus 0.06 \odot (17.5, 25.5, 36.5, 46.5)$
$\oplus 0.056 \odot (19, 28, 38, 46.5)$
$\oplus\ 0.084 \odot (19.5, 28, 39, 47.5)$
$\oplus\ 0.09 \odot (19, 27.5, 39.5, 50)$
$\oplus 0.12 \odot (18, 29.5, 40.5, 51)$
$\oplus 0.084 \odot (20.5, 30, 41, 50)$
$\oplus 0.112 \odot (19.5, 32, 42, 51)$
$\oplus 0.126 \odot (21, 30, 42, 51)$
$\oplus 0.168 \odot (20, 32, 43, 52) =$
$(18.534, 28.225, 38.773, 47.678)$

Using Table IV and formula (11), we can obtain the cumulative distribution function of the network completion time.
The cumulative distribution function of the network completion time is as follows:

$$F_{\tilde{T}}(t) = \begin{cases} 0 & t < (10,15,21,26) \\ 0.1 & (10,15,21,26) \le t < (17.5,25.5,36.5,46.5) \\ 0.16 & (17.5,25.5,36.5,46.5) \le t < (19,28,38,46.5) \\ 0.216 & (19,28,38,46.5) \le t < (19.5,28,39,47.5) \\ 0.3 & (19.5,28,39,47.5) \le t < (19,27.5,39.5,50) \\ 0.39 & (19,27.5,39.5,50) \le t < (18,29.5,40.5,51) \\ 0.51 & (18,29.5,40.5,51) \le t < (20.5,30,41,50) \\ 0.594 & (20.5,30,41,50) \le t < (19.5,32,42,51) \\ 0.706 & (19.5,32,42,51) \le t < (21,30,42,51) \\ 0.832 & (21,30,42,51) \le t < (20,32,43,52) \\ 1 & (20,32,43,52) \le t \end{cases}$$

To compute the variance of the network completion time, first, the values of $E(\tilde{T}) \otimes E(\tilde{T})$ and $E(\tilde{T} \otimes \tilde{T})$ are calculated using formulas (13) and (14).

$E(\tilde{T}) \otimes E(\tilde{T}) =$
$(343.5091, 796.6506, 1503.3455, 2273.1916)$

$E(\tilde{T} \otimes \tilde{T}) =$
$(352.557, 819.4875, 1541.5835, 2328.376)$

Then, the variance of the network completion time is obtained based on formulas (12) and (15).

$d = \max(0, (d_2 - d_1)) =$
$\max(0, (2328.376 - 2273.1916))$ =55.1843

$c = \max\left(0, min(d, (c_2 - c_1))\right) =$
$max\left(0, min(55.1843, (1541.5835 - 1503.3455))\right)$ =38.2379

$b = \max\left(0, min(c, (b_2 - b_1))\right) =$
$max\left(0, min(38.2379, (819.4875 - 796.6506))\right)$ =22.8368

$a = \max\left(0, min(b, (a_2 - a_1))\right) =$
$\max\left(0, min(22.8368, (352.557 - 343.5091))\right)$ =9.0478

$\sigma^2 = (9.0478, 22.8368, 38.2379, 55.1843)$

For example, the conditional probability function for the end node 1 is obtained using formula (16), the results of which have been shown in Table V.

TABLE V: Conditional probability function for end node 1

$P\left((\tilde{T} = \tilde{T}_{1j}) \middle| i = 1\right)$	$\tilde{T}_{1j}$
$\frac{2}{9}$	(19.5,28,39,47.5)
$\frac{1}{3}$	(21,30,42,51)
$\frac{4}{9}$	(20,32,43,52)

Using Table V and formula (17), we can obtain the conditional cumulative distribution function of the end node1.

$$F_{\tilde{T}}(t|i=1) = \begin{cases} 0 & t < (19.5,28,39,47.5) \\ \frac{2}{9} & (19.5,28,39,47.5) \le t < (21,30,42,51) \\ \frac{5}{9} & (21,30,42,51) \le t < (20,32,43,53) \\ 1 & (20,32,43,53) \le t \end{cases}$$

VIII. Conclusion and Recommendations

In this research, GERT networks have been proposed for analysis of projects in which not only the duration of activities but also their realization time is non-deterministic. An algorithm is proposed to determine the network completion time, so that by repeating the algorithm, other sub networks will be obtained. Then, the mean, variance, probabilistic function and cumulative distribution function of network completion time are calculated as a fuzzy random variable. This study is independent of the fuzzy ranking method. So, we can employ other ranking methods. Obviously, different results can be achieved using different ranking methods. Other researchers can use other ranking methods to compare the results. Also, this problem may be studied when duration of activities are random fuzzy variables. Future researchers can study the problem related to the time cost tread-off with renewable and non-renewable resources for these types of GERT networks.

References

[1] A. P. Shibanov, "Finding the distribution density of the time taken to fulfill the GERT network on the basis of equivalent simplifying transformations," *Automation and Remote Control.*, Vol. 64, No. 2, pp. 279–287, 2003.

[2] S. S. Hashemin, and S. M. T. FatemiGhomi, "A hybrid method to find cumulative distribution function of completion time of GERT networks," *J IndEng Int.*, Vol. 1, No. 1, pp. 1–9, 2005.

[3] A. A. B. Pritsker, W. W. Happ, "GERT graphical evaluation and review technique, Part I (Fundamentals)," *Journal of Industrial Engineering.*, Vol. 17, No. 5, pp. 267–274, 1966.

[4] A. A. B. Pritsker, G. E. Whitehouse, "GERT-Graphical evaluation and review technique. Part Ⅱ. Probabilistic and industrial engineering applications," *Journal of Industrial Engineering*., Vol. 17, No. 6, pp. 293-300, 1966.

[5] M. H. K. Gavareshki, "New fuzzy GERT method for research projects scheduling," *IEEE International Engineering Management Conference, Singapore: IEEE Computer Society Press*., Vol. 2, pp. 820-824, October. 2004.

[6] S. Y. Liu, S. C. Liu, and J. W. Lin, "Model formulation and development of fuzzy GERT networks," *Journal of the Chinese Institute of Industrial Engineers*., Vol. 21, No. 2, pp. 156-166, 2004.

[7] S. S. Hashemin, "Fuzzy completion time for alternative stochastic networks," *Journal of Industrial Engineering International*., Vol. 6, No. 11, pp. 17-22, 2010.

[8] H. J. Zimmermann, *Fuzzy set theory and its applications*, Third Edition. Boston, Kluwer Academic Publishers, 1996.

[9] Y. Lai and C. Hwang, *Fuzzy mathematical programming methods and applications*, New York: Springer-Verlag, 1992.

[10] A. Soltani, and R. Haji, "A project scheduling method based on fuzzy theory," *Journal of Industrial and Systems Engineering.*, Vol. 1, No. 1, pp. 70-80, 2007.

[11] C. H. Cheng, "A new approach for ranking fuzzy numbers by distance method," *Fuzzy Sets Syst.*,Vol. 95, No. 3, pp. 307-317, 1998.