An Analytical Approach for Inverse Heat Conduction Problem

Kevin Agrawal[1]
[1]Department of Mechanical Engineering, Bits Pilani, K.K. Birla Goa Campus
[1]NH 17B, Bypass Road, Zuarinagar, Sancoale, Goa, India

Abstract

The inverse method has been adopted for finding the heat flux on a plate from the temperature data. On a plate which has been heated by electron beam rastering the temperature of the plate can be found using the thermal camera, however the heat flux cannot be measured directly. For the heat flux a numerical formulation needs to be performed for which methods such as the Fourier transformation, Duhamel Theorem and the second form of Newton Leibnitz equation have to be used. Applying these methods the final formulation has been done in MATLAB and validated using ANSYS. This method could be significant for estimating the heat flux which is generated from different designs of the laser beam rastering and could find applications in various fields of science and engineering.

Keywords — *Inverse Heat Conduction problem, Heat Flux estimation, electron beam rastering, Duhamel Theorem*

I. INTRODUCTION

The general heat conduction equation in 3 dimensional space is given by the following equation: [$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q^{\cdot}}{K}=\frac{1}{\kappa}\frac{\partial T}{\partial t}$]. Depending on the initial and boundary conditions different methods to find the analytical solution for the heat conduction equation. The analytical solution commonly involves finding the temperature distribution inside the body e.g:- temperature distribution inside a pin fin subjected to convection and heat flux boundary condition. The direct and experimental measurement of heat flux at the surface of a wall subjected to heating by using conventional methods is difficult. But it can be estimated by the IHCP (Inverse Heat Conduction problem) method requiring the temperature data recordings at different points on the surface. Techniques or methods such as the Laplace transformation, separation of variables and the Duhamel's theorem can be used to find the inverse solution of such problems. Initially the solution for the inverse heat conduction problem involves finding the solution to the analytical problem, which is the temperature distribution inside the body. This analytical solution is also known as the solution to the direct problem. Subsquently the inverse problem involves using the solution to the direct problem to determine the inverse solution using different methods. Several methods have been proposed by Jaler and Duda for solving the inverse heat conduction problems, such as the Finite Element Balance Method, Finite Difference Method and the Boundary Element Method. Here an alternative method has been proposed that can be used for solving the inverse heat conduction problem. This formulation involves using the Laplace transformation and the Duhamel Theorem for finding the analytical solution or the temperature distribution for the forward problem. Subsequently the Newton Leibnitz method and integration by parts have been used for finding the solution to the inverse problem. The solution to the inverse problem involves determining the heat flux distribution from the transient temperature data.

II. MATHEMATICAL MODEL

The heat conduction transient equation in 1-D is $\frac{\partial^2 T}{\partial x^2}=\frac{1}{\kappa}\frac{\partial T}{\partial t}$ $(0<x<l)$. For the mathematical model consider an isotropic, homogeneous, semi infinite solid of constant thermal properties.

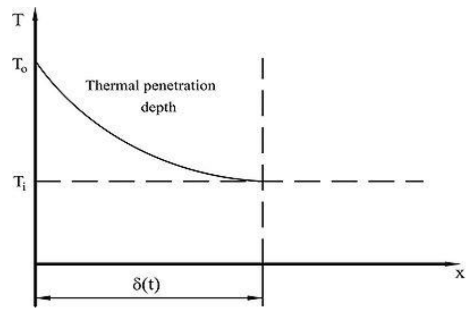

Fig. 1 Thermal Penetration Depth Inside the Wall

The semi infinite solid approximation is generally used for the simplification of the problems and can be used when the depth of the wall is greater than the thermal penetration depth as shown in the figure 1. The initial distribution in the interior is assumed to be $T_\infty = 0$ and the boundary conditions for the problem are assumed to be as follows:-

$q(t)'' = -k\frac{\partial T}{\partial x}$ at $x = 0$
(1)

$T(0,t) = T_0$ at $x = \infty$
(2)

Here the heat flux boundary condition is time dependent and uniformly rises to the steady value instead of a step change. Hence the boundary conditions assumed over here are time dependent. In these cases the Laplace transformation cannot be applied directly because the Laplace transformation cannot be applied to time dependent boudnary conditions. Hence now the Duhamels theorem will be emplobyed to find the solution to the time dependent boundary condition. The equation of Duhamel's theorem is given by:-

$$T(r,t) = \frac{\partial}{\partial t}\int_{T=0}^{t} \emptyset(r,t-\tau,\tau)d\tau$$
(3)

Here $T(r,t)$ is the solution of the time dependent boundary condition and $\emptyset(r,t-\tau,\tau)$ is the solution of the time independent boundary condition. Now here if $\emptyset(r,t-\tau,\tau) = f(\tau)\emptyset(r,t)$. then the equation (3) can rewritten as:-

$$T(r,t) = \int_{T=0}^{t} f(\tau)\,\frac{\partial\emptyset(r,t-\tau)}{\partial t}d\tau$$
(4)

So first for the direct problem, the solution of the time independent problem will be solved using the Laplace Transformation and then the Duhamel theorem will be used to find the solution to the time varying problem by using the equation (4) mentioned above.

A. *Formulation using Laplace Transformation*

The Laplace transformation of the heat conduction equation and the 2 boundary conditions are shown below in the Table 1:-

TABLE I

	Equation	Laplace Transformation
Heat Conduction eqn	$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}$	$\frac{\partial^2 \bar{v}}{\partial x^2} - \frac{p}{\kappa}\bar{v} = -\frac{T_i}{\kappa}$
Heat Flux B.C. at x = 0	$Q = -k\frac{\partial T}{\partial x}$	$\frac{Q}{p} = -k\frac{\partial \bar{v}}{\partial x}$
Constant temperature B.C. at x = ∞	$T = T_i$	$\bar{v} = \frac{T_i}{p}$

Note that, here the initial temperature T_i is assumed to be 0 and hence the transformed heat conduction equation is a 2nd order homogeneous ODE. The solution for this equation shall be found by assuming $\bar{v} = ce^{sx}$ and then substituting it in the heat conduction equation to give:- $s^2 = \frac{p}{\kappa}$. Here for simplification assume, $\frac{p}{\kappa} = q^2$ and hence $s = +q, -q$. Hence now $\bar{v} = c_1e^{qx} + c_2e^{-qx}$ and upon substituting $\bar{v}$ in the constant temperature boundary condition at x =∞, $c_1 = 0$. Hence since $c_1 = 0, \bar{v} = c_2e^{-qx}$ and substituting this equation in the heat flux boundary condition, $\frac{Q}{p\times k} = c_2$. Since $c_1 = 0$, $\frac{Q}{p\times k} = c_2$ and $\bar{v} = c_1e^{qx} + c_2e^{-qx}$. Combining these equation, $\bar{v} = \frac{Q}{p\times k}e^{-qx}$. Now using the table of Laplace transforms the solution for the time independent boundary conditions is:-

$$\emptyset(r,t-\tau,\tau) = 2F_s\frac{\left(\frac{\kappa t}{\pi}\right)^{0.5}}{K}\exp\left(\frac{-x^2}{4\kappa t}\right) - \frac{Qx}{K}erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)$$
(5)

Note that there are several methods that can be used for finding the solution for time independent boundary conditions. Instead of the Laplace Transformation the method of separation of variables could have been used as well.

B. *Formulation using Duhamel Theorem*

The solution for the time independent problem has been found using the Laplace Transformation. Now the next step is to apply the Duhamel theorem for finding the analytical solution for the time dependent boundary condition and hence to the forward problem. The Duhamel theorem is given by the equation (4):-

$$T(r,t) = \int_{T=0}^{t} f(\tau)\,\frac{\partial\emptyset(r,t-\tau)}{\partial t}d\tau$$
(4)

and from (5), the equation is:-

$$\emptyset(r,t-\tau,\tau) = 2F_s\frac{\left(\frac{\kappa t}{\pi}\right)^{0.5}}{K}\exp\left(\frac{-x^2}{4\kappa t}\right) - \frac{Qx}{K}erfc\left(\frac{x}{2\sqrt{\kappa t}}\right)$$

The solution has been found at x = 0 i.e. the boundary where the heat flux boundary condition has been applied and following the Duhamel formulation i.e. upon substituting equation (5) in (4) the solution is:-

$$T(0,t) = \frac{\kappa^{0.5}}{K\pi^{0.5}}\left(\int_0^t F_s(t-\tau)/\tau^{0.5}\right)d\tau$$
(6)

C. *Formulation using the Newton Leibnitz theorem*

In the previous section the analytical solution for the direct problem has been found and the solution the inverse problem shall be solved. The 2nd form of Newton Leibnitz on equation is given by the expression:-

$$\frac{d}{dt}\left(\int_{b(t)}^{a(t)} f(x,t)dx\right) = \int_{b(t)}^{a(t)}\frac{\partial f}{\partial t}dx + f(b(t),t).\dot{b}t - f(a(t),t).\dot{a}t$$

Now differentiating equation (6) w.r.t "t" on both sides and using the Newton Leibnitz rule :-

$$\frac{\partial}{\partial t}T(0,t) = \frac{\kappa^{0.5}}{K\pi^{0.5}}\frac{\partial}{\partial t}\left(\int_0^t F_s(t-\tau)/\tau^{0.5}\right)d\tau$$

$$\frac{\partial}{\partial t}T(0,t) - \frac{\kappa^{0.5}}{K\pi^{0.5}}\left(\int_0^t \frac{\partial}{\partial t}F_s(t-\tau)/\tau^{0.5}\right)d\tau = \frac{\kappa^{0.5}}{K\pi^{0.5}}\frac{F_s(0)}{t^{0.5}}$$

$$t^{0.5}\times\frac{\partial}{\partial t}T(0,t) - \frac{\kappa^{0.5}}{K\pi^{0.5}}\left(\int_0^t \frac{\partial}{\partial t}\frac{F_s(t-\tau)}{\tau^{0.5}}\right)d\tau\times t^{0.5} = \frac{\kappa^{0.5}}{K\pi^{0.5}}F_s(0)$$

$$t^{0.5}\times T(t) - \frac{1}{2}\int\frac{T(0,t)}{t^{0.5}}d\tau - t^{0.5}\times T(\tau) = \frac{\kappa^{0.5}}{K\pi^{0.5}}F_s(0)\times t$$

$$F_s(0,t) = \frac{K}{(4\pi\kappa)^{0.5}}\left(\int_0^t\frac{(T_s(t)-T_s(\tau))}{(t-\tau)^{1.5}}d\tau + 2\left(\frac{(T_s(t)-T_s(\tau))}{t^{0.5}}\right)\right)$$
(7)

Hence the equation (7) gives us the inverse solution to the problem which shall now be formulated in MATLAB and validated using ANSYS.

III. VALIDATION USING ANSYS

For the validation we have considered a cuboid having the dimension of $(5mm \times 50mm \times 3mm)$. In addition the material is assumed to be aluminium with material properties mentioned below:-

- $\rho = 2700\ kg/m^3$
- $K = 237\ W/mK$
- $C = 910\ J/kgK$
- $\kappa = 9.64 * 10^{-5} m^2/sec$

An isothermal boundary condition of $T(t) = 20$ is applied at y = 50mm boundary and the heat flux is applied at the boundary, y = 0mm with the variation as follows:-

TABLE II

Time(seconds)	0-1 sec	1-2 sec
Heat flux ($\frac{W}{m^2}$)	$0\ \frac{W}{m^2}$	$5\times10^6\ \frac{W}{m^2}$

The initial temperature is assumed to be $T_i = 0$ and the substeps assumed to be as 100 which will be a good approximation for the given mesh size and the given material properties. Generally the maximum number of sub steps is determined by the approximation, $ITS = \Delta^2/4\alpha$ where Δ is the conducting length of an element in the direction of heat transfer, α is the thermal diffusivity and $\frac{1}{ITS}$ is the maximum number of sub steps. In our problem we have $\Delta = 1mm$ and $\alpha = 9.64\times10^{-5}m^2/sec$. Hence the maximum number of sub steps which can be considered for a good approximation is 386.1 Subsequently for the formulation in MATLAB the steps for finding the inverse solution are mentioned below:-

- Extract the temperature T (t)
- Divide time into sub steps N.
- Take the first sub step i.e. 0 to time t_1
- Perform the integration from time 0 to t_1
- Obtain heat flux, $F_s(t_1)$
- Repeat the steps 3, 4, 5 for subsequent time steps i.e t_1 tot_N.
- Store F (t_1 tot_N) in an array
- Plot F (t) vs t

The temperature data has been obtained by the ANSYS which has been used for calculating the heat flux from MATLAB. The following heat flux variation and temperature plots were obtained from ANSYS.

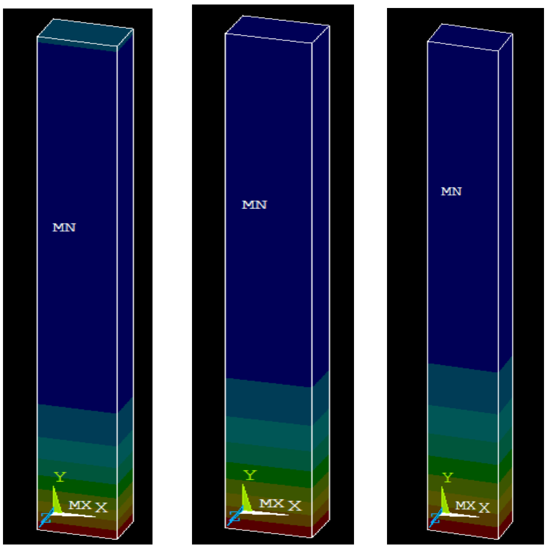

Fig. 2 Temperature Plot At 1.5, 1.75 and 2 Seconds Respectively

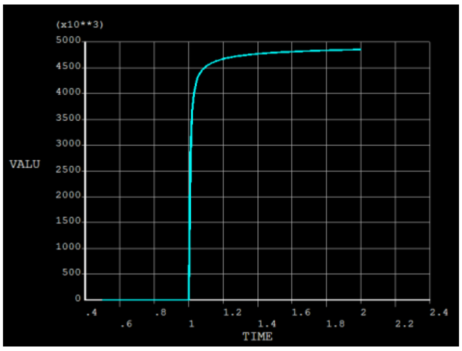

Fig. 3 Heat Flux Variation With Time at the Heat Flux Boundary (ANSYS)

As seen in the figure 2, the thermal penetration depth is shown at times of 1.5, 1.75 and 2 seconds and is less than the wall depth. Hence for this case the semi-infinite solid approximation can be safely assumed to be valid. Now the temperature vs time data from the figure (3) will be given as input to MALAB to find the heat flux. Following the formulation the heat flux plot from MATLAB has been shown below in figure 4.

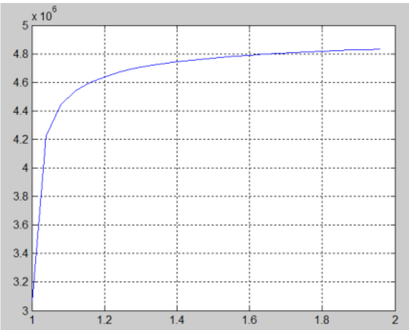

Fig. 4 Heat Flux Variation with Time at the Heat Flux Boundary (MATLAB)

By looking at the plot we can get a good idea and an estimate of the final converging heat flux value. The heat flux obtained at t = 2 sec by ANSYS was 4.855×10^6 and by MATLAB was 4.8331×10^6.

IV. CONCLUSIONS

The analytical solution of the direct problem was determined considering the semi-infinite solid approximation for a problem involving a time dependent heat flux and isothermal boundary conditions. The semi infinite solid approximation was also proved to be valid by the analysis performed in ANSYS. For finding the inverse solution we need to formulate the direct problem and then find the solution to the inverse problem. The direct solution was determined by the Laplace transformation and the Duhamel theorem. The solution to the inverse problem was determined using the Newton Leibnitz equation and the formulation of the problem was done in MATLAB. The formulation was validated using ANSYS and comparing the heat flux data calculated by the inverse formulation in MATLAB, the 2 results were found to be coherent and through MATLAB we can get a good estimate of the final value of heat flux. Since now the method is proved to be valid this method can be used in experimental setups as well. However while performing experiments on the electron beam rastering setups, the radiation heat losses, emissivity and the errors while obtaining the data from the IR camera have to be taken into account. Once accounted for, the method could be significant for estimating the heat flux which is generated from different designs of the laser beam rastering. Hence as per our requirement we can design the laser beam rastering pattern, following which the temperature data is extracted and then use the formulation in MATLAB to calculate the heat flux. This method could find vast applications in all fields of science and engineering.

ACKNOWLEDGMENT

This project was completed at the Institute for Plasma Research, Gandhinagar. I would like to thank Dr. Samir S Khirwadkar and Mr. Vinay Menon for their constant support throughout the duration of the project. I would also like to acknowledge the assistance by Mr. Rajamannar Swamy.

REFERENCES

[1] Taler, J. and Duda, P., Solving Direct and Inverse Heat Conduction Problems, Springer, Berlin (2006).

[2] Ozisik, M. N. and Orlande, R. B. H., Inverse Heat Transfer, Taylor and Francis, New York (2000).

[3] Beck, J. V., Blackwell, B. and St. Clair, C. R., Inverse Heat Conduction, III Posed Problems, A Wiley Interscience Publication, New York (1985).

[4] Tervola, P., A Method to Determine the Thermal Conductivity from Measured Temperature Profiles, Int. J. Heat Mass Transfer, 32:1425-1430 (1989).

[5] Incropera and Dewitt, Fundamentals of Heat and Mass Transfer, Seventh Edition

[6] H.S. Carslaw and J.C. Jaeger, Conduction of Heat in Solids, Oxford (1973)

[7] B.K. Bein and E.R. Muller, Derivation of the power deposition on limiters and divertot from time and space resolved thermographical measurements, Journal of Nuclear Materials

[8] Tariq Muneer, Jorge Kubie. Thomas Grassie, Heat Transfer : A Problem Solving Approach, Volume 1

[9] Lesnic, D., Elliott, L. and Ingham, D. B., The Solution of an Inverse Heat Conduction Problem Subject to the Specification of Energies", Int. J. Heat Mass Transfer, 41: 25-32 (1998).