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Periodic Solution of the Differential Equations of Machine Units

Venelin Jivkov[#1]

[#]*Theory of Mechanisms and Machines Dept., Technical University of Sofia*
8, "Kliment Ohridski" Blvd., 1756 Sofia, Bulgaria

Abstract

The paper proposes a new matrix algorithm and an iterative procedure for searching a periodic solution of the differential equation of machine units with one degree of freedom subjected to forces that depend on speed and position. To search the solution, an approach for calculating the Fourier series's coefficients for each iteration of a variant of Lyapunov's equation is proposed. For illustration, an example of a machine unit with one degree of freedom is presented. The periodic solution is in excellent agreement with similar examples in the scientific literature.

Keywords — *matrix form, periodic solution in harmonics, steady-state motion, nonlinear equation.*

I. NOMENCLATURE

φ – angular coordinate**;**

$\varphi^{\circ} = d\varphi/dt$ – angular velocity;

Φ – geometrical period, $' = d/d\varphi$ – first geometrical derivative;

$T = 1/2\, J_r\, \varphi^{\circ 2}$ – kinetic energy of machine unit;

$J_r(\varphi) = J_r(\varphi + \Phi), \forall\varphi$ – reduced mass momentum of inertia;

$M_r(,^{\circ}) = M_r(\varphi + \Phi, \varphi^{\circ})$ – reduced momentum of external forces;

T_k – the subscript denotes the iteration number of energy.

II. INTRODUCTION

The steady-state motion of the machine unit is described by the following periodical solution $\varphi^{\circ}(\varphi) = (\varphi^{\circ} + \Phi); \forall\varphi$ of the differential equation

$$J_r \varphi^{\circ} \varphi^{\circ\prime} + 0.5 J_r\, \varphi^{\circ 2} = M_r(\varphi, \varphi^{\circ}) \text{ or} \quad (1)$$

$$d/d\varphi(T) = M_r(\varphi, \varphi^{\circ}) \quad (2)$$

The present case does not deal with the known partial solution in which equation (1) or (2) permits analytical integration. The problem in the mathematical aspect, in general, is considered in the literature by the following basic methods [2], [4], [8]:

- harmonic balance, respectively harmonic linearization;
- averaging with a small increment of the parameter;
- iterative methods with integrated operators * Lyapunov, Tricomi, Loshchinin, without introducing a small parameter.

In the present work, a computational procedure is proposed based on the methods of Lyapunov, Loshchinin, Tricomy [1], [2], [3], [4], [7] in which the desired periodical function is presented in Fourier series. Final results are obtained by harmonic balance, wherein the harmonics' coefficients are determined by matrix calculations, which is a substantial advantage in computing terms.

The functional relation $M_r(\varphi, \varphi^{\circ})$ for the reduced momentum of external force for a wide range of machine's units is represented as polynomial by the degrees of the velocity $\varphi^{\circ I}$ with periodical coefficients [1], [3] like

$$M_r(\varphi, \varphi^{\circ}) = \sum \mathrm{f}_i(\varphi)\, \varphi^{\circ I}, (0 \le I \le n), \forall\varphi \quad (3)$$

III. THEORETICAL CONSIDERATION

The mathematical operations are presented below to realize the iteration procedure presented in a matrix form for solving the differential equation (1).

A. *Derivation of Harmonic Function in Fourie Series*

The first derivative of the function $\mathrm{f}(\varphi) = \mathrm{f}(\varphi + \Phi)$, i.e.

$$\mathrm{f}(\varphi) = \mathrm{f}_0 + \sum(\mathrm{f}_{cs} \cos s\varphi + \mathrm{f}_{ss} \sin s\varphi), (1 \le s \le m) \quad (4)$$

is as follows $\mathrm{f}(\varphi) = \sum(-\mathrm{s}\, \mathrm{f}_{cs} \sin s\varphi + s\, \mathrm{f}_{ss} \cos s\varphi)$.

The matrix-vector [f] with dimensions (2m+1) presented by the coefficients of (4) is:

$$[\mathrm{f}] = [\mathrm{f}_0, \mathrm{f}_{c_1}, \mathrm{f}_{c_2}, ...,, \mathrm{f}_{c_m}, \mathrm{f}_{s_1}, \mathrm{f}_{s_2}, ..., \mathrm{f}_{s_m}]^{\text{trans.}} \quad (5)$$

The coefficients of the first derivative of (4) can be obtained from the following matrix equation

$$[\mathrm{f}'] = [[\mathrm{D}]].[\mathrm{f}], \quad (6)$$

where $[[\mathrm{D}]] = \mathrm{diag}\{0, -1, -2, ..., -m, 1, 2, ..., m\}$ and

$$[\mathrm{f}'] = [0, -\mathrm{f}_{c_1}, 2\mathrm{f}_{c_2}, ..., \mathrm{m}\,\mathrm{f}_{c_m}, \mathrm{f}_{s_1}, 2\mathrm{f}_{s_2}, ..., \mathrm{m}\,\mathrm{f}_{s_m}]^{\text{trans.}}. \quad (7)$$

B. *Multiplication of Two Harmonic Series*

For example $h(\varphi+\Phi)=\mathrm{f}(\varphi+\Phi)\,g(\varphi+\Phi)$, where

$$g(\varphi)=g_0+\sum(g_{cs}\cos s\varphi+g_{ss}\sin s\varphi)\,, \quad (8)$$

in matrix form can be present as

$$[h]=[[\mathrm{f}]].[g]\,, \quad (9)$$

where the matrix-vector $[g]$ presents the harmonics of (8), or

$$[g]=[g_0, g_{c_1}\cos\varphi,...,g_{c_m}\cos m\varphi, g_{s_1}\sin\varphi,...,g_{s_m}\sin m\varphi]^{\mathrm{trans.}}. \quad (10)$$

The square matrix $[[\mathrm{f}]]$ is

$$[[\mathrm{f}]]=\begin{bmatrix} \mathrm{f}_0 & \mathrm{f}_{c_1}\cos\varphi & \mathrm{f}_{c_2}\cos 2\varphi & ... & \mathrm{f}_{c_m}\cos m\varphi & \mathrm{f}_{s_1}\sin\varphi & \mathrm{f}_{s_2}\sin 2\varphi & ... & \mathrm{f}_{s_m}\sin m\varphi \\ \mathrm{f}_{c_1}\cos\varphi & \mathrm{f}_{c_2}\cos 2\varphi & ... & \mathrm{f}_{c_m}\cos m\varphi & \mathrm{f}_{s_1}\sin\varphi & \mathrm{f}_{s_2}\sin 2\varphi & ... & \mathrm{f}_{s_m}\sin m\varphi & \mathrm{f}_0 \\ \mathrm{f}_{c_2}\cos 2\varphi & ... & \mathrm{f}_{c_m}\cos m\varphi & \mathrm{f}_{s_1}\sin\varphi & \mathrm{f}_{s_2}\sin 2\varphi & ... & \mathrm{f}_{s_m}\sin m\varphi & \mathrm{f}_0 & \mathrm{f}_{c_1}\cos\varphi \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ \mathrm{f}_{s_m}\sin m\varphi & \mathrm{f}_0 & \mathrm{f}_{c_1}\cos\varphi & \mathrm{f}_{c_2}\cos 2\varphi & ... & \mathrm{f}_{c_m}\cos m\varphi & \mathrm{f}_{s_1}\sin\varphi & ... & \mathrm{f}_{s_{m-1}}\sin(m-1)\varphi \end{bmatrix}. \quad (11)$$

C. *Root Square of Periodic Function*

From the dependence

$$\mathrm{f}(\varphi)=\mathrm{y}^2(\varphi)\,, \quad (12)$$

follows the differential equation

$$2\,\mathrm{f}(\varphi)\,\mathrm{y}'(\varphi)-\mathrm{f}'(\varphi)\,\mathrm{y}(\varphi)=0\,, \quad (13)$$

which in matrix form is recorded by

$$\{2[\mathrm{f}(\varphi)][[\mathrm{D}]]-[[\mathrm{D}]][\mathrm{f}(\varphi)]\}[\mathrm{y}(\varphi)]=0\,, \quad (14)$$

If the harmonic series

$$\mathrm{y}(\varphi+\Phi)=\mathrm{y}_0+\mathrm{y}_{c_1}\cos\varphi+...+\mathrm{y}_{s_m}\sin m\varphi\,, \quad (15)$$

is represent as

$$\mathrm{y}(\varphi)=\mathrm{y}_0(1+\overset{*}{\mathrm{y}}_{c_1}\cos\varphi+...+\overset{*}{\mathrm{y}}_{s_m}\sin m\varphi)\,, \quad (16)$$

where $\mathrm{y}_{c_m}=\mathrm{y}_0\,\overset{*}{\mathrm{y}}_{c_s}, \mathrm{y}_{s_s}=\mathrm{y}_0\,\overset{*}{\mathrm{y}}_{c_i}$ $(1\le \mathrm{s}\le \mathrm{m})$ relation (14) can be rearing as

$$\begin{aligned}&\{2[\mathrm{f}(\varphi)]_{-0}[[\mathrm{D}]]_{-0}-[[\mathrm{D}]]_{-0}[\mathrm{f}(\varphi)]_{-0}\}[\mathrm{y}^*(\varphi)]_{-0}=\,,\\&=2[[\mathrm{D}]]_{-0}[\mathrm{f}(\varphi)]_{-0}\end{aligned} \quad (17)$$

where the index "-0" signifies that the columns and rows in determinants are corresponding to y_0 absent.

Coefficients of the harmonics in the function $\mathrm{y}(\varphi)$ - (16) can be determined by means of the rule of Parseval (the equality of the average values)

$$\mathrm{y}_0^2\left\{1+\frac{1}{2}\sum(\overset{*2}{\mathrm{y}}_{c_s}+\overset{*2}{\mathrm{y}}_{s_s})\right\}=\mathrm{f}_0\,, \quad (1\le \mathrm{s}\le \mathrm{m})\,. \quad (18)$$

D. *Iteration Procedure*

Sequential iterations for determining the periodic components $\mathrm{T}(\varphi)$ can be represented by differential equation concerning the increase of the kinetic energy of the system in two adjacent iterations

$$\Delta\mathrm{T}'_{k,k+1}+\lambda_2\Delta\mathrm{T}_{k,k+1}=\mathrm{M}_r(\varphi,\overset{\circ}{\varphi}_k)-\mathrm{T}'_k(\varphi)\,. \quad (19)$$

where $\mathrm{T}'_k(\varphi)$ is the momentum of inertial forces? The relation (19) for each iteration is a linear nonhomogeneous first-order linear equation with guaranteed periodic solution, which leads to integrated operators of Tricomi and Loshchinin [5], [6]. The solution can be obtained by the harmonic balance method, calculating the harmonics' coefficients with the above mathematical dependencies in matrix form.

The steps to solve the problem are as follows:

- set the initial approximation - $\mathrm{T}(\varphi)$;
- determining the geometric derivative $\mathrm{T}'(\varphi)$;

$$[\mathrm{T}']=[[\mathrm{D}]][\mathrm{T}(\varphi)]\,;$$

- calculated $\overset{\circ}{\varphi}{}^2$ as a product of two periodic functions;
- determining of square root equation (17) - $[\varphi^\circ]$;
- calculating the reduced momentum $\mathrm{M}_r(\varphi,\varphi^\circ)$;
- calculating $\Delta\mathrm{T}$;

$$\{[[\mathrm{D}]]-\lambda_2[[E]]\}[\Delta\mathrm{T}]=[\mathrm{M}_r-\mathrm{T}']$$

- *if* $[\Delta\mathrm{T}/T]\le\varepsilon$ - the end ;
- *it is not* - a new approximation, and the procedure is repeated from the beginning. The conditions for convergence of the iteration depends on the limitation of the first derivative of the reduced momentum $\mathrm{M}_r(\varphi,\varphi^\circ)$ in the interval [5]

$$\mathrm{T}_{\min}\le\mathrm{T}\le\mathrm{T}_{\max} \text{ or}$$

$$\lambda_2\le d\,\mathrm{M}_r(\varphi,\mathrm{T})/d\,\mathrm{T}\le-\lambda_1,\ 0<\lambda_1\le\lambda_2\,. \quad (20)$$

TABLE I

Values of the kinetic Energy Harmonics Coefficients

	T_0	T_1	T_2		T_6	T_7
C_6						
C_5						
C_4						
C_3					-0,00003	-0,00003
C_2			-0.00638		-0,00703	-0,00703
C_1		-0.67568	-0.75677		-0,75891	-0,75891
C_0	7,0	7,05773	7.04910		7.04419	7,04419

S_1		0,46811	0,43768		0,42749	0,42749
S_2			-0.00016		-0,00194	-0,00194
S_3					-0,00007	-0,00007
S_4						
S_5						
S_6						

IV. NUMERICAL EXAMPLE

Let's consider a machine unit with the following - parameters [5]:

$$M_r = 2 + \sin\varphi - 0.04\varphi^{\circ 4}[\mathrm{N\,m}],$$

$$\mathrm{J} = 1[\mathrm{kg\,m}^2], \quad \lambda_1 = 0.4, \quad \lambda_2 = 0.69282.$$

In Table 1, the calculated values of the kinetic energy harmonics coefficients up to the seventh iteration are shown. They are estimated using the proposed matrix algorithm for the periodic solution (1), (2).

The presented results match to the fifth decimal point with those quoted in [5].

V. CONCLUSIONS

The proposed algorithm in matrix form for searching a periodic solution in Fourier series can be applied to calculate the harmonic coefficients' values in functional differential equations with periodic functions.

The major result of the proposed matrix algorithm for searching for a periodical solution consists of the dependencies (9), (17), and (19) here derived.

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REFERENCES

[1] Kolovskiy M. "*Dynamics of Machines"* (in Russian) Л. 1989 "Машиностроение. "

[2] Kolovskiy M. Non-linear "*Dynamics of Active and Passive Systems of Vibration Protection.",* Springer Verlag (in English) 1999.

[3] Wulfson I., Kolovskiy M. *"Non-Lineare Problems in Machin's Dynamics"* (in Russian) Изд. Машиностроение, Л. 1968.

[4] Kolovskiy M. "*Advanced Theory of Mechanisms and Machines.*" Springer Verlag (in English) 1999.

[5] Артоболевский И. Лощинин В. Динамика Машинных *"Агрегатов на Предельных Режимах Движения"* (in Russian) Изд. Наука М. 1977.

[6] Tricomi F. Periodische "*Losungen einer Differentialgleihnung erster Ordnung".* Verh. Int.Math. Kongress , Zurich №2, 1932.

[7] Kauderer H. *Nichtlineare Mechanik*, B/G/H, Springer Verlag 1958.

[8] ide Julius Popoola, Oladele Joshua Oladejo and Charity Segun Odeyemi, "Modelling and Simulation of Armature-Controlled Direct Current Motor using MATLAB" SSRG International Journal of Electrical and Electronics Engineering 2.3 (2015): 12-18.