Optimal Control for a Distributed Parameter System with Time-Delay, Non-Linear Using the Numerical Method. Application to One-Sided Heat Conduction System

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Abstract

This paper presents a solution of an optimal control problem for a parabolic-type distributed parameter systems with time delay, non-linear governed by a heat-conduction equation. The system is applied to a specific one-sided heat-conduction system in a heating furnace to control temperature for a slab following the most accurate burning standards [2], [6]. The target of problem is to find an optimal control signal so that the error between the distribution of real temperature of the object and the desired temperature is minimum after a given period of time t_f [2], [6], [9]. After solving the problem, building the algorithms and establishing the control programs, we have proceeded to run the simulation programs on a slab of Samot and a slab of Diatomite to test calculting programs.

Keywords: *optimal control, distributed parameter systems, delay, non-linear, numerical method.*

I. INTRODUCTION

Optimal control for distributed parameter system is applied in many fields such as heat treatments, composting the magnetic materials, steel rolling, etc.

In some previous technologies [2], [6], [7], the heating process was carried out in a burning furnace with FO heavy oil, such as burning in steel rolling or in the processes of manufacture of aluminum, glass. In this case, the transfer function of the furnace is the delayed inertia, and the relationship between the temperature of furnace is the parabolic-type partial differential equation with the boundary condition of type 3. If we consider the optimal control problem for the "most accurate burning process", the control object is now distributed parameter system with time-delay.

With this problem, some authors have been interested in and solved by variational method, using the Pontryagin's maximum principle or numerical method as in [1], [2], [6]. However, in some other technologies, the furnace is an electrical furnace, it means burned by a resistor wire such as heat treatments of mechanical parts, composting the magnetic materials, etc. Hence, the transfer function of resistor furnace is also the first order inertia system with time delay in the form of:

$$W(s) = \frac{Y(s)}{X(s)} = \frac{k.e^{-\tau s}}{(Ts+1)}$$

But, at this case k is the coefficient depending on the temperature in the furnace. Actually, by identifying a real resistor furnace, k varies considerably, for example in a resistor furnace with a temperature range of 0-500^{0}C. (This will be demonstrated later).

If the optimal control problem for the "most accurate burning process" is considered, then this is the optimal control problem for the object with distributed, delayed, nonlinear parameters. The nonlinearity of k that makes the solution of the problem becoming complex.

Thus, in order to solve the problem, this paper divided k into several values, then applies the Laplace transform and the numerical method to give an explicit solution to heat transfer problem.

II. THE PROBLEM OF OPTIMAL CONTROL

A. *The object model*

As a typical distributed parameter system, a one-dimensional heat conduction system is considered. The process of one-sided heating of object, which shaped like a retangular box in a furnace is described by the parabolic-type partial differential equation, as follows in [2], [6], [8], [9]:

$$a\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t} \tag{1}$$

where $q(x,t)$, the temperature distribution in the object, is the output needing to be controlled, depending on the spatial coordinate x with $0 \le x \le L$ and the time t with $0 \le t \le t_f$, a is the temperature-conducting factor (m^2/s), L is the thickness of object (m), t_f is the allowed burning time (s)

The initial and boundary conditions are given in [2],[6],[9].

$$q(x,0) = q_0(x) = const \tag{2}$$

$$\lambda \left.\frac{\partial q(x,t)}{\partial x}\right|_{x=0} = \alpha\left[q(0,t) - v(t)\right] \tag{3}$$

$$\left.\frac{\partial q(x,t)}{\partial x}\right|_{x=L} = 0 \tag{4}$$

with α as the heat-transfer coefficient between the furnace space and the object ($W/m^2.^0C$), λ as the heat-conducting coefficient of material ($W/m.^0C$), and $v(t)$ as the temperature of the furnace respectively (^0C).

The temperature $v(t)$ of the furnace is controlled by voltage $u(t)$, the temperature distribution $q(x,t)$ in the object is controlled by means of the fuel flow $v(t)$, this temperature is controlled by voltage $u(t)$. Therefore, the temperature distribution $q(x,t)$ will depend on voltage $u(t)$.

The relationship between the provided voltage for the furnace $u(t)$ and the temperature of the furnace $v(t)$ is usually the first order inertia system with time delay as in [1], [2], [6], [9].

$$T.\dot{v}(t) + v(t) = k.u(t-\tau) \tag{5}$$

where T is the time constant, τ is the time delay; k is the static transfer coefficient; $v(t)$ is the temperature of the furnace and $u(t)$ is the provided voltage for the furnace (controlled function of the system).

However, in expression (5), k is the changing coefficient depending on the temperature in the furnace, it means k is a function of temperature v. So, the static transfer coefficient can be expressed by the equation: $\bar{k} = k(v)$, so $\bar{k}$ is a nonlinear coefficient. Thus, the expression (5) can be expressed by the equation:

$$T.\dot{v}(t) + v(t) = \bar{k}.u(t-\tau) \tag{6}$$

However, when the coefficient $\bar{k}$ is nonlinear, it is difficult to find a solution, on the other hand, we can not to apply the Laplace transform. Therefore, the paper will perform the linearization of coefficients $\bar{k}$ into N values: $\bar{k}_1, \bar{k}_2, \bar{k}_3, \ldots, \bar{k}_N$. In which, we assume that coefficients $\bar{k}_1, \bar{k}_2, \bar{k}_3, \ldots, \bar{k}_N$ are constants.

B. The objective function

The problem is set out as follows: we have to determine a control function $u(t)$ with ($0 \le t \le t_f$) in order to minimize the temperature difference between the distribution of desired temperature $q^*(x)$ and real temperature of the object $q(x,t_f)$ at time $t = t_f$. It means at the end of the heating process to ensure temperature uniformity throughout the whole material:

$$J_c = \int_0^L \left[q^*(x) - q(x,t_f)\right]^2 dx \to \min \tag{7}$$

The constrained conditions of the control function is:

$$U_1 \le u(t) \le U_2 \tag{8}$$

with U_1,U_2 are the lower and upper limit of the supply voltage respectively (V). This problem is called the most accurate burning problem.

III. IDENTIFICATION OF RESISTOR FURNACE MODEL

1. Resistor furnace model

Fig 1. Resistor furnace model

2. Transfer function of resistor furnace

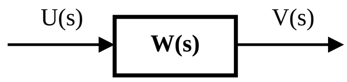

Fig 2. Transfer function of resistor furnace

According to Ziegler-Nichols, the resistor furnace model can be expressed in the form of a transfer function as the first order inertia system having delayed as follows:

$$W(s) = \frac{V(s)}{U(s)} = \frac{k.e^{-\tau s}}{(Ts+1)} \tag{9}$$

The block diagram of the identifying system is shown in Fig .3.

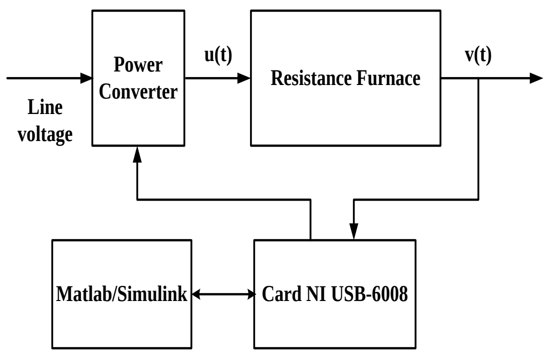

Fig 3. Diagram of data collecting system

The system uses NI USB-6008 interface card to output the voltage signal to the furnace control and collects the temperature signal in the furnace. The voltage passed from the computer to the NI USB-6008 card, through the DAC to the voltage converter, thereby changing the voltage supplied to the furnace. The temperature in the furnace is measured by the thermocouple, after via the standardization kit is also taken to the NI USB-6008 Card, which converts into digital signals and sends temperature data to computer for the identification of the system model.

Put into the furnace a step voltage: u(t) = 220.1(t), after a period of time Δt=4500(s), the furnace temperature reach to $v_f \approx 500(^0C)$. We obtained at the output the temperature response of the resistor furnace as in Fig. 4.

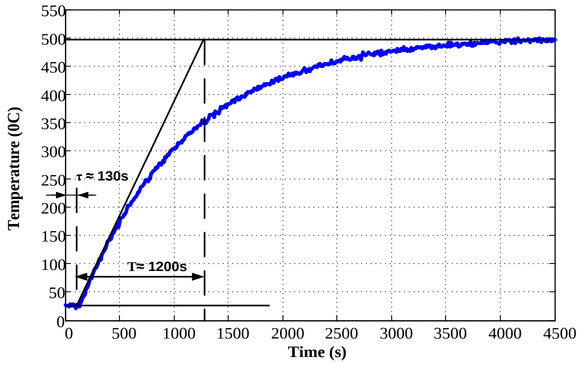

Fig 4. Temperature response of the resistor furnace

According to Ziegler-Nichols, we can determine time constants of the furnace as: *T≈1200(s); τ≈130(s)*

The static transfer coefficient is determined:

$$k = \frac{v_f}{U} \approx \frac{500}{220} \approx 2{,}27$$

where v_f is the set temperature, U is the supplied voltage to the furnace.

Comment

When the furnace temperature reached to v_f, we have $k = const$, however, the temperature of the furnace *v(t)* depends on the time *t*, as the furnace temperature changes from ambient temperature v_0 to set temperature v_f, then *k* also changes depending on temperature, it means $k \neq const$. Thus, it is very difficult to determine exactly the coefficient *k* at each time *t*. To analyze the coefficient *k*, that depends on the temperature *v(t)*, theoretically we can perform as:

Keep set temperature v_f = const, call the voltage changing intervals are *Δu* (V), the changing intervals of furnace temperature are *Δv* (^{0}C). Putting into the furnace that the voltage changes in the form of steps, after a period of time *Δt*, the voltage increases a quantity *Δu* to the voltage *u = 220* (V), the static transfer coefficient of the furnace $\bar{k}$ corresponds to each time intervals *Δt* can be calculated:

$$\bar{k}_i = \Delta v_i \,/\, \Delta u_i \qquad i = 1, 2, 3, \ldots, N \tag{9}$$

From Eq. (9), we see that in the whole burning time from *0* to t_f, for each pair of values (*Δu, Δv*), we will have *N* values $\bar{k}$, so when applying to find the solution for the problem, the number of calculations will be large and extremely complex. So, the author has done as follows:

Do not supply directly the voltage 220(V) that put into the furnace the voltage changes in the form of steps, after each time intervals *Δt* = 4500 (s), if the voltage increases a quantity *Δu*, the furnace temperature will increase accordingly a quantity *Δv* to the voltage *u = 220 (V)*, the furnace temperature will reach the set temperature $v_f = 500\ (^0C)$, the test time is *t = 13500* (s). Thus, in reality, we will have many pairs of values (*Δu, Δv*) and will have many values $\bar{k}$ respectively. By identifying the real resistor furnace, we saw $\bar{k}$ need only 3 or 4 values that can be satisfied (output temperature will be reached to set temperature $v_f \approx 500\ (^0C)$, so within the set temperature range, this paper only considers 3 values of the coefficient $\bar{k}$, that is $\bar{k} = \bar{k}_1; \bar{k}_2; \bar{k}_3$.

The analysis of coefficients $\bar{k}$ into 3 values will make the solution of the problem becoming simpler.

The identification curve is shown in Fig 5.

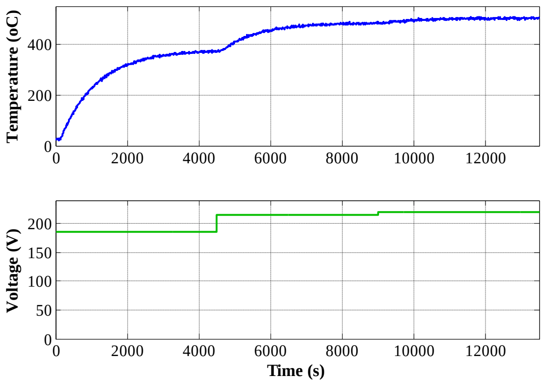

Fig 5. Experimental results of resistor furnace model identification

Comment

From the experimental characteristic curve is shown in Figure 5 and expression (9), the static transfer coefficient of the furnace $\bar{k}$ is not constant, it depends on the temperature in the furnace *v(t).*

To determine the coefficients $\bar{k}$ as shown in Fig.5, we have Fig.6.

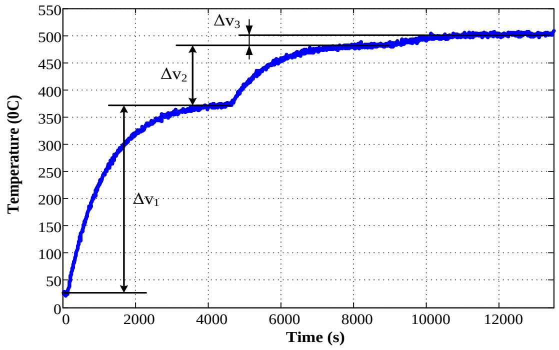

Fig 6. Temperature response to determine Δv.

From Fig.5 and Fig.6, we can determine the temperature changing ranges Δv and the voltage changing ranges Δu, the static transfer coefficients of the furnace $\bar{k}$ corresponding to each time intervals Δt are calculated as in Table 1.

Table 1. Table of coefficients $\bar{k}$

N	Δv (^0C)	Δu (V)	$\bar{k} \approx \Delta v / \Delta u$
1	350	185	1,8
2	100	30	3,3
3	25	5	5

Comment

From Fig.6 and Table 1, the static transfer coefficient of the furnace will increase as the furnace temperature increases over time. When the furnace temperature range changes from ambient temperature to about 500^{0}C, the value of $\bar{k}$ changes quite a lot.

IV. THE SOLUTION OF PROBLEM

The paper proposes a method to solve the optimal problem for the above system as follows: we divided the heating time from *0* to t_f into 3 equal time periods $\Delta t_1 = \Delta t_2 = \Delta t_3$ and call:

- $\Delta t_1 = 0 \div t_1$ corresponding to $\Delta v_1 = v_0 \div v_1$, we have $\bar{k}_1$

- $\Delta t_2 = t_1 \div t_2$ corresponding to $\Delta v_2 = v_1 \div v_2$, we have $\bar{k}_2$

- $\Delta t_3 = t_2 \div t_f$ corresponding to $\Delta v_3 = v_2 \div v_f$, we have $\bar{k}_3$

with $t_1 = t_f / 3$; $t_2 = 2t_f / 3$; t_f as the allowed burning time; v_0 as ambient temperature, v_f as the set temperature.

The process of finding the optimal solution includes 2 steps:

- S*tep 1*: Find the relationship between $q(x,t)$ and the control signal $u(t)$. Namely, we have to solve the equation of heat transfer (relationship between $v(t)$ and $q(x,t)$) with boundary condition type-3 combined with ordinary differential equation with time delay (relationship between $u(t)$ and $v(t)$)

- S*tep 2*: Find the optimal control signal $u^*(t)$ by substituting $q(x,t)$ found in the first step into the function (7), after that finding optimal solution $u^*(t)$.

We consider in the first period of time $\Delta t_1 = 0 \div t_1$ as follows:

A. *Find the relationship between $q_1(x,t)$ and the control signal $u_1(t)$*

To solve the partial differential equation (1) with the initial and the boundary conditions (2), (3), (4), we apply the Laplace transformation method with the time parameter *t*. On applying the transform with respect to *t*, the partial differential equation is reduced to an ordinary differential equation of variable *x*. The general solution of the ordinary differential equation is fitted to the boundary conditions, and the final solution is obtained by the application of the inverse transformation.

Transforming Laplace (1), we obtained:

$$a\frac{\partial^2 Q_1(x,s)}{\partial x^2} = sQ_1(x,s) \qquad (10)$$

where: $Q_1(x,s) = L\{q_1(x,t)\}$

After transforming the boundary conditions (3), (4), we have:

$$\lambda \left.\frac{\partial Q_1(x,s)}{\partial x}\right|_{x=0} = \alpha\left[Q_1(0,s) - V_1(s)\right] \qquad (11)$$

$$\left.\frac{\partial Q_1(x,s)}{\partial x}\right|_{x=L} = 0 \qquad (12)$$

From Eq. (6), assuming the delayed object satisfy the condition: $6 \le T/\tau < 10$ in [5], [9]. To solve this problem, the first order inertia system with time delay is replaced by first-order Pade appximation (Pade 1)

Transforming Laplace (6), we obtained:

$$(Ts+1)V_1(s) = \bar{k}_1.U_1(s).e^{-\tau s} \approx \bar{k}_1.U_1(s)\frac{2-\tau s}{2+\tau s}$$

(13)

where: $V_1(s) = \mathbf{L}\{v_1(t)\}$; $U_1(s) = \mathbf{L}\{u_1(t)\}$

(14)

The general solution of (1) is:

$$Q_1(x,s) = A_1(s).e^{\sqrt{\frac{s}{a}}.x} + B_1(s).e^{-\sqrt{\frac{s}{a}}.x}$$

(15)

where: $A_1(s)$; $B_1(s)$ are the parameters need to be find.

After transforming, we have the function:

$$Q_1(x,s)=\frac{U_1(s).\bar{k}_1.(2-\tau s)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)}+e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)(2+\tau s)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L}+e^{\sqrt{\frac{s}{a}}.L}\right]-\lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L}-e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \quad (16)$$

Putting

$$G_1(x,s)=\frac{\bar{k}_1.(2-\tau s)\left[e^{-\sqrt{\frac{s}{a}}.(L-x)}+e^{\sqrt{\frac{s}{a}}.(L-x)}\right]}{(Ts+1)(2+\tau s)\left\{\left[e^{-\sqrt{\frac{s}{a}}.L}+e^{\sqrt{\frac{s}{a}}.L}\right]-\lambda.\frac{\sqrt{\frac{s}{a}}}{\alpha}.\left[e^{-\sqrt{\frac{s}{a}}.L}-e^{\sqrt{\frac{s}{a}}.L}\right]\right\}} \quad (17)$$

We have: $Q_1(x,s) = G_1(x,s).U(s)$ (18)

From (18), according to the convolution theorem, the inverse transformation of (18) is given by

$$q_1(x,t) = g_1(x,t)* u_1(t)$$

We can write

$$q_1(x,t)=\int_0^t g_1(x,\tau).u_1(t-\tau)d\tau \quad (19)$$

or

$$q_1(x,t)=\int_0^t g_1(x,t-\tau).u_1(\tau)d\tau \quad (20)$$

where

$$g_1(x,t)=\mathbf{L}^{-1}\{G_1(x,s)\} \quad (21)$$

Therefore, if we know the function $g_1(x,t)$, we will be able to calculate the temperature distribution $q_1(x,t)$ from control function $u_1(t)$. To find $q_1(x,t)$ in (20), we need to find the function (21). Using the inverse Laplace transformation of function $G_1(x,s)$ we have the following result:

$$g_1(x,t)=\frac{\bar{k}_1.k_0^2\left(2+\tau k_0^2\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(2-\tau k_0^2\right)\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0^2t}+\frac{2\bar{k}_1.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+\sum_{i=2}^{\infty}\frac{2\alpha.\bar{k}_1\left(2+\tau.\Psi_i^2\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(2-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2t} \quad (22)$$

To find the temperature distribution $q_2(x,t)$ with t in the range $\Delta t_2 = t_1 \div t_2$ and the temperature distribution $q_3(x,t)$ with t in the range $\Delta t_3 = t_2 \div t_f$, we also transform the same as the case of finding $q_1(x,t)$, we finally get the following result:

$$g_2(x,t)=\frac{\bar{k}_2.k_0^2\left(2+\tau k_0^2\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(2-\tau k_0^2\right)\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0^2t}+\frac{2\bar{k}_2.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+\sum_{i=2}^{\infty}\frac{2\alpha.\bar{k}_2\left(2+\tau.\Psi_i^2\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(2-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2t} \quad (23)$$

$$g_3(x,t)=\frac{\bar{k}_3.k_0^2\left(2+\tau k_0^2\right).\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(2-\tau k_0^2\right)\left[\cos\left(\frac{k_0L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0L}{\sqrt{a}}\right)\right]}.e^{-k_0^2t}+\frac{2\bar{k}_3.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-Tk_1^2\right)\left[\cos\left(\frac{k_1L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1L}{\sqrt{a}}\right)\right]}.e^{-k_1^2t}+\sum_{i=2}^{\infty}\frac{2\alpha.\bar{k}_3\left(2+\tau.\Psi_i^2\right)\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(2-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2t} \quad (24)$$

$$k_0 = 1/\sqrt{T}\,; \qquad k_1 = \sqrt{2/\tau}$$

In the case, $e^{-\tau s}$ is replaced by Taylor approximation in [2], [6]. Eq. (6) becomes:

$$(Ts+1)V_1(s)=\bar{k}_1.U_1(s).e^{-\tau s}\approx\bar{k}_1\frac{U_1(s)}{\tau s+1} \quad (25)$$

In general case, in order to find the functions $g_\mu(x,t)$ ($\mu=1,2,3$), we also have the same transformation as in the case of Pade 1, finally we obtained the functions $g_\mu(x,t)$ according to the Taylor expansion as follows:

$$g_{\mu}(x,t)=\frac{\bar{k}_{\mu}.k_0^2.\cos\left(\frac{k_0}{\sqrt{a}}(L-x)\right)}{\left(1-\tau k_0^2\right)\left[\cos\left(\frac{k_0 L}{\sqrt{a}}\right)-\frac{\lambda k_0}{\alpha\sqrt{a}}\sin\left(\frac{k_0 L}{\sqrt{a}}\right)\right]}.e^{-k_0^2 t}+$$

$$+\frac{\bar{k}_{\mu}.k_1^2.\cos\left(\frac{k_1}{\sqrt{a}}(L-x)\right)}{\left(1-T.k_1^2\right)\left[\cos\left(\frac{k_1 L}{\sqrt{a}}\right)-\frac{\lambda k_1}{\alpha\sqrt{a}}\sin\left(\frac{k_1 L}{\sqrt{a}}\right)\right]}.e^{-k_1^2 t}+$$

$$+\sum_{i=2}^{\infty}\frac{2\alpha\bar{k}_{\mu}\cos\left(\frac{\Psi_i}{\sqrt{a}}(L-x)\right)}{\lambda\left(1-T\Psi_i^2\right)\left(1-\tau.\Psi_i^2\right)\left[\frac{\lambda+\alpha L}{\lambda.\Psi_i\sqrt{a}}\sin\left(\frac{\Psi_i.L}{\sqrt{a}}\right)+\frac{L}{a}\cos\left(\frac{\Psi_i.L}{\sqrt{a}}\right)\right]}.e^{-\Psi_i^2 t} \quad (26)$$

$$k_0=1/\sqrt{T};\quad k_1=\sqrt{1/\tau}$$

In Eq.(23), Eq.(24), Eq.(25) and Eq. (26): Ψ_i is calculated from the formula:

$$\Psi_i=\phi_i\sqrt{a}/L$$

- ϕ_i is the solution of the equation:

$$\phi.tg\phi=\alpha L/\lambda=B_i$$

- Bi is the coefficient BIO of the material.
- α is the heat-transfer factor (W/m^2.^{0}C)
- λ is the heat-conducting factor of object (W/m.^{0}C)
- L is the thickness of object (m),
- a is the temperature-conducting factor (m^2/s)
- τ is the time delay of the furnace (s)
- T is the time constant of the furnace (s)
- $\bar{k}_1;\bar{k}_2;\bar{k}_3$ are the static transfer coefficients of the furnace corresponding to periods of time Δt_1; Δt_2; Δt_3 respectively.

Conclusions:

We have solved a system of parabolic-type partial differential equation with boundary conditions of type-3 (the relationship between $v_1(t)$ and $q_1(x,t)$) combined with the ordinary differential equation with time delay (the relationship between $u_1(t)$ and $v_1(t)$).

Thus, if we are not interested in the optimal problem, we can calculate the temperature field in the object when we know the supplied voltage for the furnace (The problem knows the shell to find the cores), as follows:

The relationship between the supplied voltage for the furnace u(t) and the temperature field distribution in the object q(x,t):

$$q_i(x,t)=g_i(x,t)*u_i(t)=\int_0^t g_i(x,t-\tau).u_i(\tau)d\tau \quad (27)$$

with $i=1\div3$; $t=0\div t_f$; t_f is the allowed burning time (s).

B. Find the optimal control signal $u^*(t)$ by using numerical method

To find the $u^*(t)$, we have to minimize the objective function (7), it means:

$$J_c=\int_0^L\left[q^*(x)-q(x,t_f)\right]^2dx\to\min \quad (28)$$

where

$$q(x,t_f)=\int_0^{t_1}g_1(x,t-\tau).u_1(\tau)d\tau+\int_{t_1}^{t_2}g_2(x,t-\tau).u_2(\tau)d\tau+\int_{t_2}^{t_f}g_3(x,t-\tau).u_3(\tau)d\tau \quad (29)$$

and $q^*(x)$ is the desired temperature distribution; $q(x,t_f)$ is the real temperature distribution of the object at time $t=t_f$.

As calculated in [2], [6], [9] the integral numerial method is used by applying Simson formula to the right-hand side of the objective function (28). The L, the thickness of the object, is divided into n equal lengths (n is an even number).

Thus, the objective function is expressed as in [2], [6].

$$J_c[u^*]=L\sum_{i=0}^{n}\xi_i\left[q^*(x_i)-q(x_i,t_f)\right]^2 \quad (30)$$

where ξ_i are the weights assigned to the values of integrand at the points x_i. The values of x_i and the weights ξ_i are known for each integration formula.

If the Simpson's composite formula is used, the values of x_i and ξ_i are given by in [2], [6].

$$\left.\begin{aligned}&x_i=L_i/n;(i=0,1,\ldots,n)\\&\xi_0=\xi_n=1/3n\\&\xi_1=\xi_3=\xi_{n-1}=4/3n\\&\xi_2=\xi_4=\xi_{n-2}=2/3n\end{aligned}\right\}\ n\text{ is an even number}$$

Similarly, it is applied to the right-hand side of the equation (29). The period of time t_f is devided into three equal intervals m_1, m_2 m_3 that m_1, m_2 m_3 are an even number, too.

where m_1 is time interval from 0 to t_1

m_2 is time interval from t_1 to t_2

m_3 is time interval from t_2 to t_f

Therefore, $q(x_i,t_f)$ is caculated:

$$q(x_i,t_f)\cong t_1\sum_{j_1=0}^{m_1}\xi_{j_1}g_1(x_i,t-\tau_{j_1}).u_1(\tau_{j_1})+$$

$$+(t_2-t_1)\sum_{j_2=0}^{m_2}\xi_{j_2}g_2(x_i,t-\tau_{j_2}).u_2(\tau_{j_2})+$$

$$+(t_f-t_2)\sum_{j_3=0}^{m_3}\xi_{j_3}g_3(x_i,t-\tau_{j_3}).u_3(\tau_{j_3}) \quad (31)$$

the values of $\tau_{j_1};\tau_{j_2};\tau_{j_3}$ and $\xi_{j_1};\xi_{j_2};\xi_{j_3}$ are given by in [2], [6].

$$\left.\begin{aligned}&\tau_{j_1} = j_1 t_1 / m_1\\&\xi_0 = \xi_{m_1} = 1/3m_1\\&\xi_1 = \xi_3 = \xi_{m_1-1} = 4/3m_1\\&\xi_2 = \xi_4 = \xi_{m_1-2} = 2/3m_1\end{aligned}\right\} (j_1 = 0,1,2,\ldots,m_1)$$

$$\left.\begin{aligned}&\tau_{j_2} = j_2 t_2 / m_2\\&\xi_0 = \xi_{m_2} = 1/3m_2\\&\xi_1 = \xi_3 = \xi_{m_2-1} = 4/3m_2\\&\xi_2 = \xi_4 = \xi_{m_2-2} = 2/3m_2\end{aligned}\right\} j_2 = m_1, m_1+1,..,m_2$$

$$\left.\begin{aligned}&\tau_{j_3} = j_3 t_f / m_3\\&\xi_0 = \xi_{m_3} = 1/3m_3\\&\xi_1 = \xi_3 = \xi_{m_3-1} = 4/3m_3\\&\xi_2 = \xi_4 = \xi_{m_3-2} = 2/3m_3\end{aligned}\right\} j_3 = m_2, m_2+1,..,m_3$$

Putting

$$\begin{aligned}&c_{1ij} = t_1.\xi_{j_1}.g_1(x_i, t-\tau_{j_1}); \quad u_1(\tau_{j_1}) = u_{j_1}\\&c_{2ij} = (t_2 - t_1).\xi_{j_2}.g_2(x_i, t-\tau_{j_2}); \quad u_2(\tau_{j_2}) = u_{j_2} \qquad (32)\\&c_{3ij} = (t_f - t_2).\xi_{j_3}.g_3(x_i, t-\tau_{j_3}); \quad u_3(\tau_{j_3}) = u_{j_3}\end{aligned}$$

$$u_{j_1} = u_{j_2} = u_{j_3} = u_j\ ;\ q*(x_i) = q_i^* \qquad (33)$$

Substituting (31), (32) and (33) into (30), we obtained:

$$J_c[u^*] = L\sum_{i=0}^{n}\xi_i\left[q_i^* - \left(\sum_{j_1=0}^{m_1}c_{1ij}.u_j + \sum_{j_2=0}^{m_2}c_{2ij}.u_j + \sum_{j_3=0}^{m_3}c_{3ij}.u_j\right)\right]^2 \qquad (34)$$

The constrained conditions of the control function (Limit of the supplied voltage for the furnace) are described as follows:

$$U_1 \le u_j \le U_2 \qquad (35)$$

$$(j = m_1+m_2+m_3)$$

The performance index (34) is a quadratic function of the variables u_j with constraints (35) are linear, the problem becomes a quadratic programming problem. This problem can be obtained by using numerical method after a finite number of iterations of computation.

Although a solution of the quadratic programming problem is obtained after a finite number of iterations of computation, but its algorithm is more complicated than that of the simplex method for linear programming. If the performance index is taken as

$$J_c = \int_0^L \left|q*(x) - q(x, t_f)\right| dx \qquad (36)$$

instead of (28), the linear programming technique can be used directly. On applying the same procedure as mentioned above, the approximate performance index corresponding to (36) is written as

$$J_c \cong \bar{J}_c = L\sum_{i=0}^{n}\xi_i\left|q_i^* - \left(\sum_{j_1=0}^{m_1}c_{1ij}.u_j + \sum_{j_2=0}^{m_2}c_{2ij}.u_j + \sum_{j_3=0}^{m_3}c_{3ij}.u_j\right)\right| \qquad (37)$$

The problem of minimizing (37) under the constraints (35) can be put into a linear programming form by using known techniques [2], [6]. By introducing *2(n+1)* non-negative auxiliary σ_i and ω_i $(i=0,1,2\ldots n)$, the minimization of (37) is equivalent to the minimization of

$$\bar{J}_c' = L\sum_{i=0}^{n}\xi_i(\sigma_i + \omega_i) \qquad (38)$$

with the constraints

$$\left.\begin{aligned}&\sum_{j=0}^{m_1+m_2+m_3} c_{ij}u_j - q_i^* = \sigma_i - \omega_i\\&\sigma_i \ge 0, \quad \omega_i \ge 0\end{aligned}\right\}(i = 0,1,\ldots,n) \qquad (39)$$

and $U_1 \le u_j \le U_2$

$(j = m_1+m_2+m_3)$

Thus, with any u_j, the minimum value of (38) is attained by setting $\omega_i = 0$ if

$\sum_{j=0}^{m_3} c_{ij}u_j - q_i^*$ is non-negative and $\sigma_i = 0$ if

$\sum_{j=0}^{m_3} c_{ij}u_j - q_i^*$ is negative. Then, clearly

$$\min \bar{J}_c = \min \bar{J}_c'$$

Thus, we can replace the solution of (34) with the constraint (35) by minimizing the problem (38) with the constraint (39).

By using the simple method in [2], [4], the optimal solution of (38), (39) can be reached after a finite number of iterations.

C. Calculate the temperature of the furnace v(t) and the temperature distribution in the object q(x,t)

1) Calculate the temperature of the furnace v(t)

We know that *v(t)* and *u(t)* have the relation:

$$T.\dot{v}(t) + v(t) = \bar{k}.u(t-\tau) \qquad (40)$$

or

$$\dot{v}(t) = \frac{\bar{k}.u(t-\tau) - v(t)}{T} = f(v,u) \qquad (41)$$

Based on improved Euler formula, we have:

$$v(j+1) = v(j) + (l_1 + l_2 + l_3).f(u, v(j))$$

$$v(j+1) = v(j) + \left[\frac{t_1}{m_1} + \frac{t_2}{m_2} + \frac{t_f}{m_3}\right] \cdot \left[\frac{\bar{k}.u(j) - v(j)}{T}\right] \quad (42)$$

where

$l_1 = t_1 / m_1$; $l_2 = (t_2 - t_1) / m_2$; $l_3 = (t_f - t_2) / m_3$; $t_1 = t_f / 3$; $t_2 = 2t_f / 3$; t_f is the allowed burning time (s); $\bar{k}$ can get: $\bar{k} = \left(\bar{k}_1 + \bar{k}_2 + \bar{k}_3\right) / 3$

with m_1, m_2, m_3 are the number of time intervals corresponding to the time intervals Δt_1, Δt_2 ,Δt_3 T is the time constant of the furnace.

After transforming, we get

$$v(j+1) = \frac{k.(l_1 + l_2 + l_3).u(j) + v(j)[T - (l_1 + l_2 + l_3)]}{T} \quad (43)$$

or

$$v(j) = \frac{k.(l_1 + l_2 + l_3).u(j-1) + v(j-1).[T - (l_1 + l_2 + l_3)]}{T} \quad (44)$$

with $j = m_1 + m_2 + m_3$

So, when knowing $u^*(t)$ we can calculate $v(t)$ from Eq. (44).

*2) **Calculate the temperature distribution in the object q(x,t)***

To calculate $q(x,t)$ when knowing $u^*(t)$, we use the privious calculated results. Here also use the numerical method [2], [3], [4], [6].

From Eq. (31), we have

$$q(x_i, t_j) = \int_0^{t_1} g_1(x_i, t_j - \tau).u_1(\tau)d\tau + \int_{t_1}^{t_2} g_2(x_i, t_j - \tau).u_2(\tau)d\tau + \int_{t_2}^{t_f} g_3(x_i, t_j - \tau).u_3(\tau)d\tau \quad (45)$$

$i = 0 \div n$; $t = 0 \div t_f$; n is the number of layers of space. According to trapezoidal formula [3], [4]. After calculating, we obtained

$$q(x_i, t) \approx j_1 l_1 \sum_{\varepsilon=0}^{j_1\delta} \xi_\varepsilon . g_1(x_i, j_1 l_1 - \tau_\varepsilon).u_1(\tau_\varepsilon) + j_2 l_2 \sum_{\varepsilon=0}^{j_2\delta} \xi_\varepsilon . g_2(x_i, j_2 l_2 - \tau_\varepsilon).u_2(\tau_\varepsilon) + j_3 l_3 \sum_{\varepsilon=0}^{j_3\delta} \xi_\varepsilon . g_3(x_i, j_3 l_3 - \tau_\varepsilon).u_3(\tau_\varepsilon) \quad (46)$$

V. SIMULATION RESULTS

After building the algorithms and establishing the control programs, we have proceeded to run the simulation programs to test calculating programs.

A. *The simulation for a slab of Samot*

- The physical parameters of the object

  $\alpha = 60$ (W/m^2.^{0}C); $\lambda = 0.955$ (W/m.^{0}C)

  $a = 4.86*10^{-7}$ (m^2/s); $L = 0.03$ (m)

- The parameters of the furnace

  $T = 1200$ (s); $\tau = 130$ (s);

  $\bar{k}_1 = 1.8$; $\bar{k}_2 = 3.3$; $\bar{k}_3 = 5$

- The desired temperature distribution $q^* = 300^0C$
- The period of heating time $t_f = 4200$ (s)
- Limit the temperature of furnace $u(t) \leq 500^0$C
- Limit the temperature of slab surface:

  $q(0,t) \leq 350\ ^0$C

- Limit under voltage: $U_1=125$ (V)
- Limit upper voltage: $U_2=205$ (V)

With these parameters, the coefficient Bi is calculated as follows:

$$Bi = \alpha.L/\lambda = 60.0,03 / 0.955 \approx 1,9$$

Thus, the slab of Samot is a thick object because the coefficient Bi is greater than 0.5. Having $6 \leq T/\tau < 10$ To calculate the optimal heating process, we choose $n = 6$ and $m_1=m_2=m_3=m=42$. After the simulation, we have results like in Figure 7 and Figure 8.

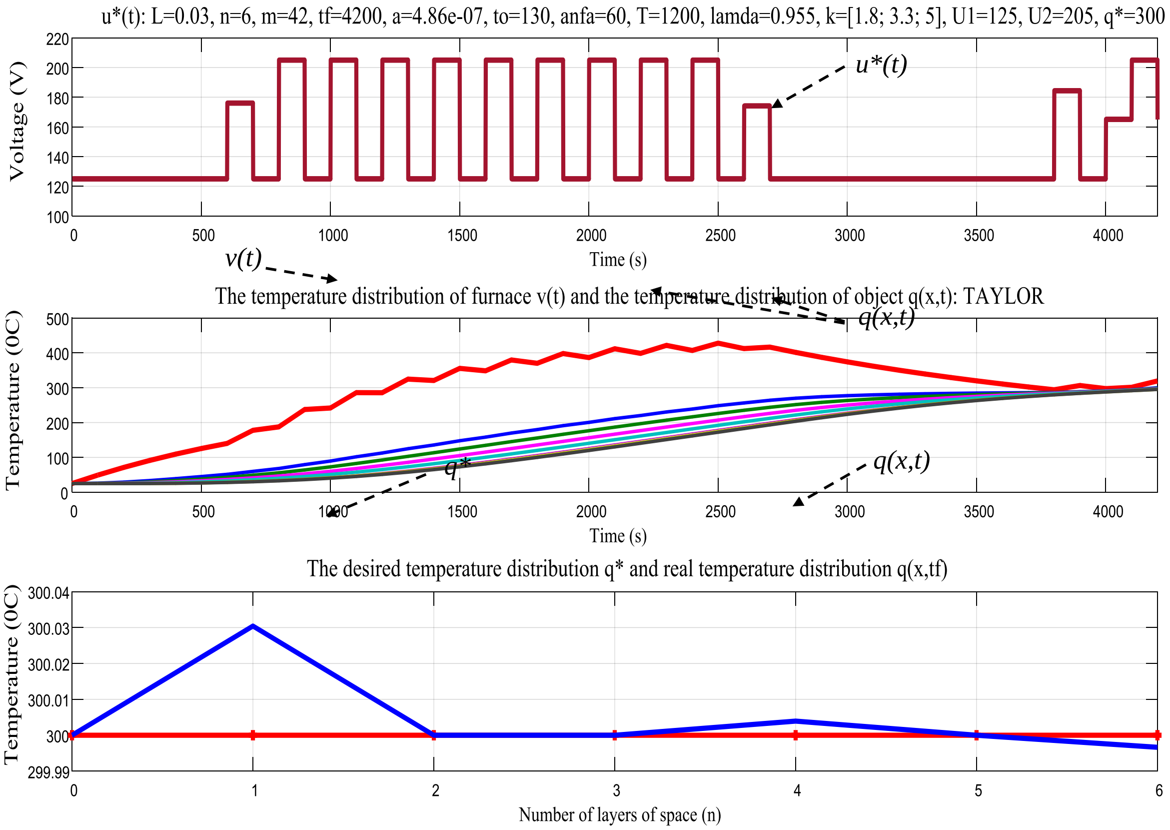

Fig 7. The optimal heating process for a slab of Samot (following Taylor Approximation, with e = 3.2077e^{-06})

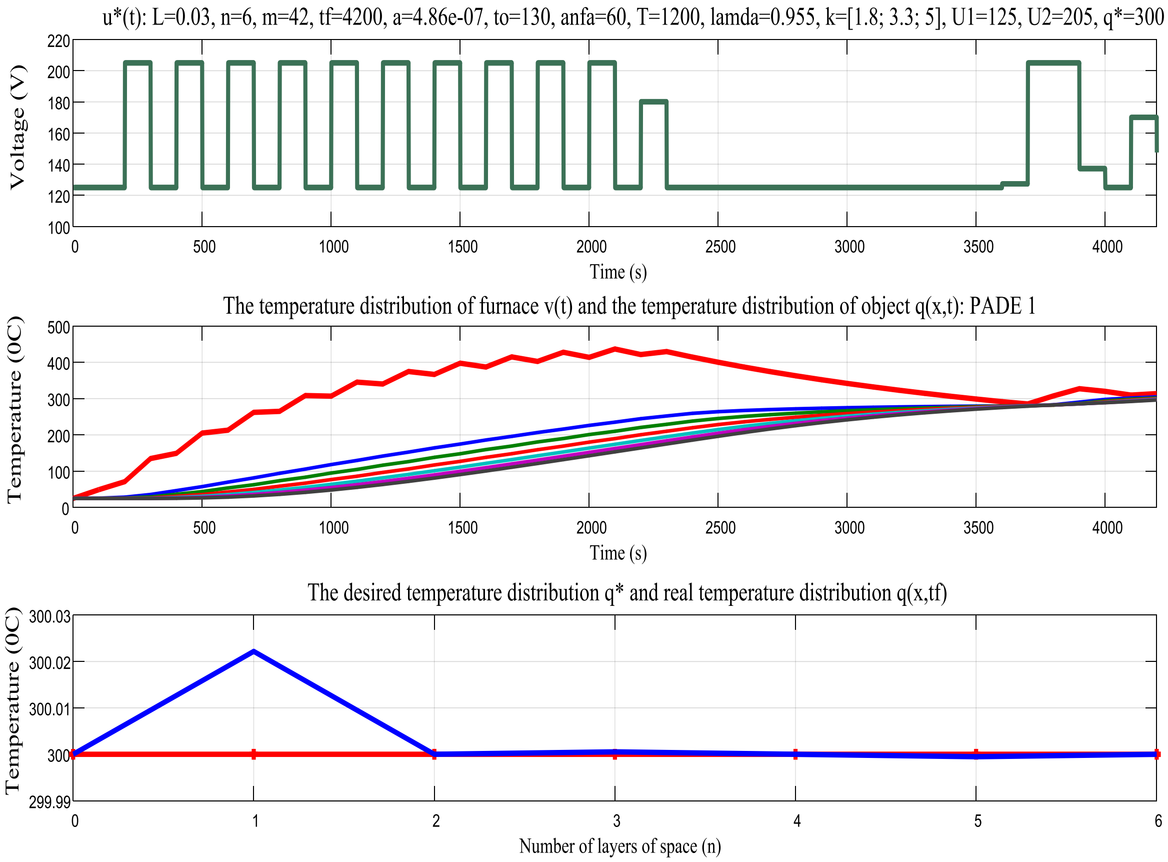

Fig 8. The optimal heating process for a slab of Samot (following Pade' Approximation, with e = 1.6441e^{-07})

B. The simulation for a slab of Diatomite

- The physical parameters of the object

  $\alpha = 60\ (w/m^{2\ 0}C);\ \lambda = 0.2\ \ (w/m.^{0}C)$
  $a = 3.6*10^{-7}\ (m^2/s);\ L = 0.04\ (m)$

- The parameters of the furnace

  $T = 1200\ (s);\ \tau = 130\ (s)$

  $\bar{k}_1 = 1.8\ ;\ \bar{k}_2 = 3.3\ ;\ \bar{k}_3 = 5$

- The desired temperature distribution $q^* = 400\ ^0C$
- The period of heating time $t_f = 4400$ (s)
- Limit the temperature of furnace $u(t) \le 750^0$C
- Limit the temperature of slab surface

  $q(0,t) \le 550\ ^0$C

- Limit under, upper voltage: $U_1=125$ (V);$U_2=220$ (V)

With these parameters, the coefficient *Bi* is calculated as follows:

$$Bi = \alpha.L/\lambda = 60.0{,}04 / 0.2 \approx 12$$

Thus, the slab of Diatomite is a very thick object because the coefficient *Bi* is greater than 0.5. Having $6 \le T/\tau < 10$

To calculate the optimal heating process, we choose $n = 10$ and $m_1=m_2=m_3=m=100$. After the simulation, we have results like in Figure 9 and Figure 10.

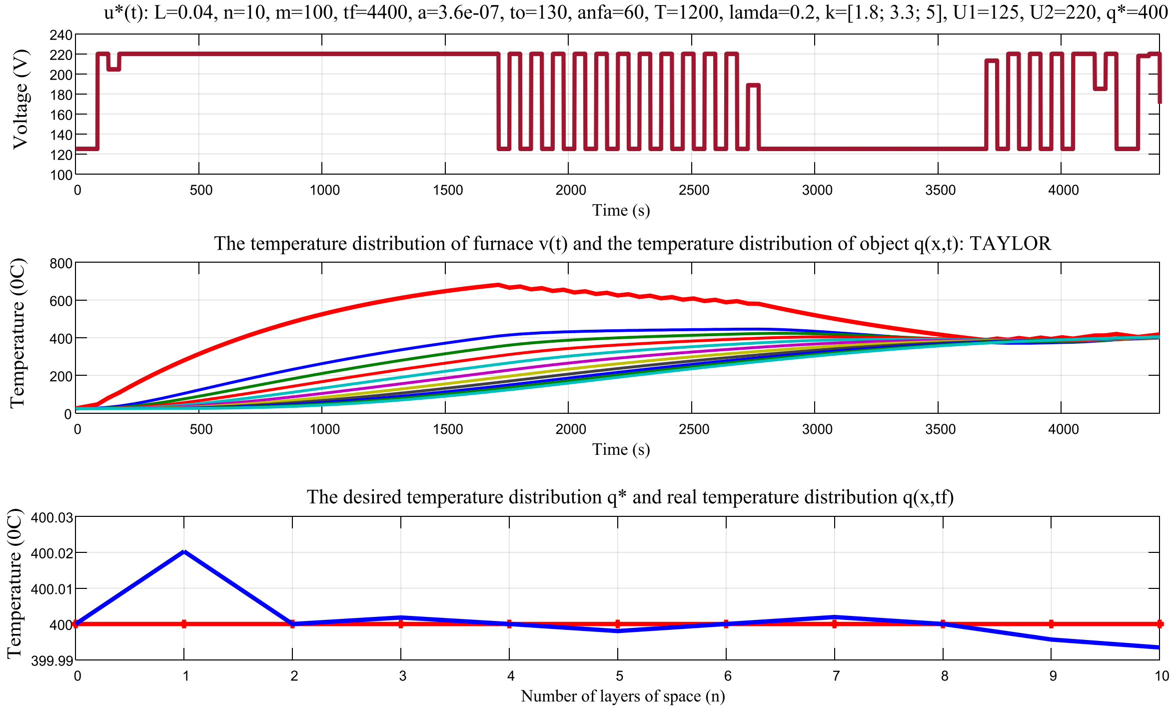

Fig 9. The optimal heating process for a slab of Diatomite (following Taylor Approximation, with *e* = 1.2319e^{-06})

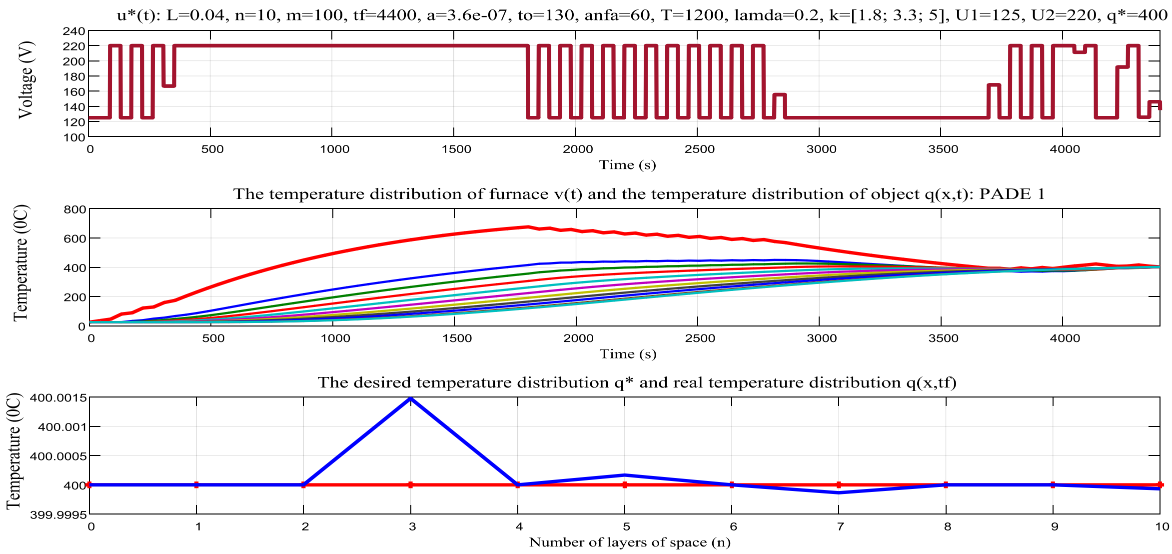

Fig 10. The optimal heating process for a slab of Diatomite (following Pade' Approximation, with *e* = 5.9423e^{-09})

The simulation results are shown on Fig.7 to Fig.10.

where $u^*(t)$ is the optimal control signal (optimal voltage) of the furnace; $v(t)$ is the temperature of the furnace; $q(x,t)$ is temperature distribution of the slab, including the temperature of the two surfaces and the temperature of the inner layers of the slab.

C. Comment

In Fig. 7 and Fig. 8, we can see that at the time $t=t_f=4200(s)$, the temperature distributions of the layers in a slab of Samot $q(x,t_f)$ are all approximately equal $q^*=300^0C$. Therefore, the optimal solution has been testified. Taylor approximation has e = 3.2077e^{-06}, whereas Pade 1 has e = 1.6441e^{-07} (with e as the error of objective function J_c).

In Fig. 9 and Fig. 10, we also can see that at the time $t=t_f=4400(s)$, the temperature distributions of the layers in a slab of Diatomite $q(x,t_f)$ are all approximately equal $q^*=400^0C$. Thus, the optimal solution has been testified. Taylor approximation has e = 1.2319e^{-06}, Pade 1 has e = 5.9423e^{-09}.

It means that, if delayed object satisfies the condition $6 \leq T/\tau < 10$ [5], [9], the first-order Pade approximation will have higher accuracy.

VI. CONCLUSIONS

The paper has solved a system consitting of parabolic-type partial differential equation with boundary condition type-3 combined with a time-delayed ordinary differential equation. Namely, we have proceeded to identify a real resitsor furnace in order to determine exactly transfer function of furnace, as well as have analyzed the change of static transfer coefficient depending on temperature in the furnace.

An optimal solution for a distributed parameter system with time-delay, nonlinear has been defined by using a numerical method. Algorithms and optimal calculating program have been accuracy. Then, we have proceeded to run the simulations on a slab of Samot and a slab of Diatomite in order to test the algorithms once again.

However, in this paper, we only simulate on Matlab software to verify the solution and the results of the calculation. In the next paper, we will conduct experiments on specific specimens in order to test the simulation results.

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