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Large Deflections of an Inextensible, Flexible Elastica Subjected to Any Load

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Abstract
This work concerns the numerical analysis of large deflections of an inextensible, flexible elastica subjected to any load. In many cases, elastica can represent, for example, a prismatic beam of a longitudinal cross-section of a textile structure. These objects under the influence of loads often experience large deflections. Also, the solution can be used for non-prismatic beams with various end conditions, and a numerical solution is presented to obtain exact solutions.

Keywords: *elastica, bending rigidity, large deflections, numerical methods, beam theory.*

I. INTRODUCTION

The large deflection of beams has been investigated by Bisshopp and Drucker [1] for a point load on a cantilever beam. Timoshenko and Gere [2] developed the solution for the axial load. The problem of the column with load through fixed point was also presented by Timoshenko and Gere [2]. Virginia Rohde [3] developed the solution for the uniform load on the cantilever beam. John H. Law [4] solved it for a point load at the tip of the beam and a uniform load combined. In this paper, the general solution developed for a prismatic beam and, in some cases, for non-prismatic. However, numerical integration may be needed along with solving compatibilities equations for the constants of integrations. A more general and preferable numerical solution for a non-prismatic beam is also given using only many point loads acting with an angle on the beam with a moment on the node representing the approximate load. This point load can take any direction on the beam bending in the *x*-*x*-direction or *y*-*y*-direction of the moment of inertia. Thus the load is to be resolved to *x*-*x*-direction and *y*-*y*-direction of the moment of inertia in each orthogonal deflection given two non-linear differential equations. By solving each non-linear differential equation, the orthogonal deflection components can be obtained. An approximate attempt has been investigated by Scott and Caver [5] for all problems in which the moment can be expressed as a function of the independent variable. Jong-Dar Yau [6] presented a solution for Closed-Form Solution of Large Deflection for a Guyed Cantilever Column Pulled by an Inclination Cable. Large deflections of bending structures were also considered by Szablewski using the elastica theory (in works [7]–[10]).

II. MATERIALS AND METHODS

The presented analysis is based on solving the non-linear differential equation of Bernoulli-Euler beam theory.

$$\frac{y''(x)}{\left(\sqrt{1+[y'(x)^2]}\right)^3}=\frac{M(x)}{EI(x)}, \tag{1}$$

Where $M(x)$ is the bending moment in the direction that corresponds to the moment of inertia $I(x)$, E is the modulus of elasticity, and y is the orthogonal deflection.
It is assumed the modulus of elasticity is constant, and the bending does not alter the length of the beam. Only one closed-form solution is investigated for one case of the non-linear differential equation. In many ways, $M(x)$ is not known until the final deflection is known. It will be assumed that $M(x)$ is known in the equations.
In general, for a non-prismatic beam, $I(x)$ is not constant along beam length. The value of $I(x)$ is a function of x and is different for different values of the x coordinate. The paper considers a simpler case for a prismatic beam. It was assumed that $I(x) = I$ is constant for the whole meam. Let us denote further that $y(x) = y$.

In this case, assume $\frac{M(x)}{EI}=f(x)$ a function of x only. Thus

$$\frac{y''}{\left(\sqrt{1+[y'^2]}\right)^3}=f(x), \tag{2}$$

Let $y'=\tan\varphi \rightarrow \sqrt{1+[y'^2]}=\frac{1}{\cos\varphi}$ and $y''=\frac{\varphi'}{\cos^2\varphi}$.

φ is a new variable, then substitute φ in Equation 2 yields
$\varphi'\cos\varphi=f(x)\rightarrow\int\cos\varphi\, d\varphi=\int f(x)\,dx+C_1$

Or $\sin\varphi = \int f(x)\,dx + C_1$.

$$\cos\varphi = \pm\sqrt{1-\sin^2\varphi} = \pm\left[1-\left(\int f(x)\,dx + C_1\right)^2\right]^{\frac{1}{2}}.$$

Thus

$$y' = \frac{\sin\varphi}{\cos\varphi} = \pm\frac{\int f(x)\,dx + C_1}{\sqrt{1-\left[\int f(x)\,dx + C_1\right]^2}}. \qquad (3)$$

$$y = \pm\int \frac{\int f(x)\,dx + C_1}{\sqrt{1-\left[\int f(x)\,dx + C_1\right]^2}}\,dx + C_2. \qquad (4)$$

C_1 and C_2 are constants of integration. This, of course, the solution Scott and Carver approximated as an infinite series, not realizing it can be expressed in a closed-form. Equation 4 gives an integrated solution where if the moment is approximated by a curve, it can give a better approximation than small deflection equations approximations. It is seen that if the denominator of Equation 4 is approximated as a unity, it will give the standard solution for small deflection approximations.

III. NUMERICAL SOLUTION

This example is to demonstrate the solution for a cantilever beam. Other boundary conditions for beams are similar. First, divide the beam into segmental beams of each length l_i, and on each node of the segment, insert the equivalent load P_i and moment Q_i (see Fig. 1).

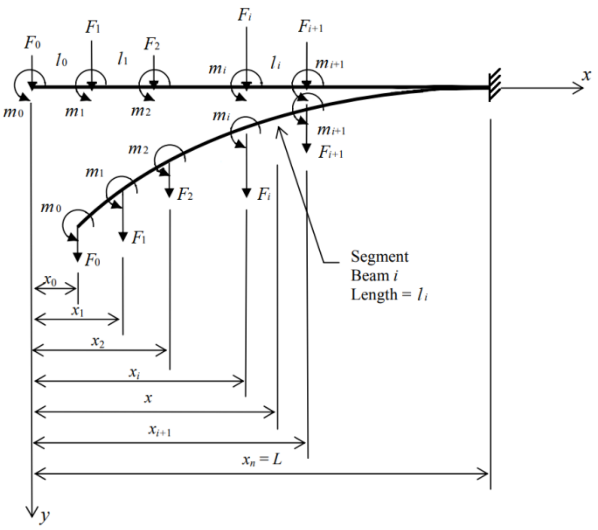

Figure 1: The infinitesimal section of elastica

The moment on the segment beam at x_i is

$$M_0 = F_0(x-x_0) + m_0 \text{ for } x_0 \le x < x_1$$

$$M_1 = F_0(x-x_0) + m_0 + F_1(x-x_1) + m_1 \text{ for } x_1 \le x < x_2$$

...

$$M_i = \sum_{j=0}^{i} F_j(x-x_j) + m_j \text{ for } x_i \le x < x_{i+1}$$

...

$$M_{n-1} = \sum_{j=0}^{n-1} F_j(x-x_j) + m_j \text{ for } x_{n-1} \le x < x_n$$

………………………………………………………… 5)

Where all x_i is unknown.

It is assumed that $I(x) = I =$ const. Therefore

$$f_i(x) = \frac{M_i}{EI} = \frac{1}{EI}\left[\sum_{j=0}^{i} F_j(x-x_j) + m_j\right]. \qquad (6)$$

Substitute Equation 6 in Equation 3 and find the slope on the segmental beam i yield

$$y_i'(x) = \frac{\frac{1}{EI}\left[\sum_{j=0}^{i}\frac{1}{2}F_j(x-x_j)^2 + m_j(x-x_j)\right] + C_{1i}}{\sqrt{1-\left\{\frac{1}{EI}\left[\sum_{j=0}^{i}\frac{1}{2}F_j(x-x_j)^2 + m_j(x-x_j)\right] + C_{1i}\right\}^2}}$$

for $x_i \le x \le x_{i+1}$ ………………………………………… (7)

Let's apply the condition of continuity

$$y_{i-1}'(x_i) = y_i'(x_i) \qquad \text{at } x = x_i \qquad (8)$$

At $x = L$, where L is the length of the beam at $x = x_n = L$

$$y_n'(x_n) = y_n'(L) = 0, \qquad (9)$$

where n is the number of beam segments, and n+1 is the total number of beam segments. Using Equation 8 and Equation 9 in Equation 7, we have

$$\frac{1}{EI}\left[\sum_{j=0}^{n-1}\frac{1}{2}F_j(L-x_j)^2 + m_j(L-x_j)\right] + C_{1n-1} = 0$$

and

$$C_{1n-1} = C_1 = -\frac{1}{EI}\left[\sum_{j=0}^{n-1}\frac{1}{2}F_j(L-x_j)^2 + m_j(L-x_j)\right],$$

since $F_n = m_n = 0$ …………………………………… (10)

When applying Equation 8 for all i we have

$$C_{10} = C_{11} = C_{12} = \ldots = C_{1n-1} = C_1$$

Ultimately

$$y_i'(x) = \frac{\frac{1}{EI}\left[\sum_{j=0}^{i}\frac{1}{2}F_j(x-x_j)^2 + m_j(x-x_j)\right] + C_1}{\sqrt{1-\left\{\frac{1}{EI}\left[\sum_{j=0}^{i}\frac{1}{2}F_j(x-x_j)^2 + m_j(x-x_j)\right] + C_1\right\}^2}} \qquad (11)$$

Now let's apply the assumption that the beam segment is inextensible. The length of segment l_i is

$$l_i = \int_{x_i}^{x_{i+1}} \sqrt{1+\left[y_i'(x)\right]^2}$$

After the transformations, we get

$$l_i = \int_{x_i}^{x_{i+1}} \frac{dx}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\sum_{j=0}^{i} \frac{1}{2} F_j (x - x_j)^2 + m_j (x - x_j) \right] + C_1 \right\}^2}} . \quad (12)$$

To simplify the equation, assume the increments are small enough such that the slope throughout the interval of $x_i \leq x \leq x_{i+1}$ is the same

$$y_i'(x_i) \cong y_i'(x_{i+1}) , \quad (13)$$

Therefore

$$l_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \Delta x_i \sqrt{1 + \left[y_i'(x_i) \right]^2} . \quad (14)$$

$$l_i = \frac{x_{i+1} - x_i}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\sum_{j=0}^{i} \frac{1}{2} F_j (x_i - x_j)^2 + m_j (x_i - x_j) \right] + C_1 \right\}^2}} . \quad (15)$$

Thus at $i = 0$ yields

$$l_0 = \frac{x_1 - x_0}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\frac{1}{2} F_0 (x_0 - x_0)^2 + m_0 (x_0 - x_0) \right] + C_1 \right\}^2}} .$$

$$l_0 = \frac{x_1 - x_0}{\sqrt{1 - (C_1)^2}} . \quad (16)$$

At $i = 1$

$$l_1 = \frac{x_2 - x_1}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\frac{1}{2} F_0 (x_1 - x_0)^2 + m_0 (x_1 - x_0) \right] + \frac{1}{EI} \left[\frac{1}{2} F_1 (x_1 - x_1)^2 + m_1 (x_1 - x_1) \right] + C_1 \right\}^2}} ,$$

$$l_1 = \frac{x_2 - x_1}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\frac{1}{2} F_0 (x_1 - x_0)^2 + m_0 (x_1 - x_0) \right] + C_1 \right\}^2}} . \quad (17)$$

At $i = 2$

$$l_2 = \frac{x_3 - x_2}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\frac{1}{2} F_0 (x_2 - x_0)^2 + m_0 (x_2 - x_0) \right] + \frac{1}{EI} \left[\frac{1}{2} F_1 (x_2 - x_1)^2 + m_1 (x_2 - x_1) \right] + C_1 \right\}^2}} . \quad (18)$$

Using x_1-x_0 from Equation 16 and x_2-x_1 from Equation 17 in Equation 18, (for a given C_1) x_3-x_2 is found, where x_2-x_0 = (x_2-x_1) + (x_1-x_0).

Thus, if guessing C_1, then find $x_{i+1} - x_i$ can be found since the denominator of Equation 15 is always known from previous equations. And since

$$\sum_{i=0}^{n} (x_{i+1} - x_i) = x_n - x_0 = L - x_0 ,$$

x_0 can be found.

Therefore for a given C_1 x_0, x_1, x_2, …, x_{n-1} can be solved, then proceed by checking the end slope of Equation 9 or Equation 10. If it is not satisfied, update C_1 with numerical analysis until all the variables are found. The deflection from Equation 4 yields

$$y_i(x) = \int \frac{\frac{1}{EI} \left[\sum_{j=0}^{i} \frac{1}{2} F_j (x - x_j)^2 + m_j (x - x_j) \right] + C_1}{\sqrt{1 - \left\{ \frac{1}{EI} \left[\sum_{j=0}^{i} \frac{1}{2} F_j (x - x_j)^2 + m_j (x - x_j) \right] + C_1 \right\}^2}} dx + C_{2i} ,$$

for $x_i < x < x_{i+1}$. …………………………………………(19)

To find C_{2i}, let's apply the condition of continuity

$y_n(x_n) = y_n(L) = 0$ and find C_{2n-1},

$y_{n-1}(x_{n-1}) = y_n(x_{n-1})$ and find C_{2n-2}

etc.

The solution is found numerically.

IV. RESULTS OF CALCULATIONS

Numerical calculations were performed for the following parameters.

L = 0.6 m,

E = 199,95 GPa,

Circular cross-section: fi = 0.01 m, $I = 4.9087 \cdot 10^{-10}$ m^4

Fig. 2 shows the results of the calculations for the problem mentioned.

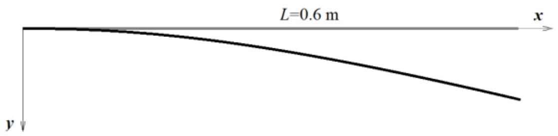

Figure 2: Calculation result for an example task

In order to verify the correctness of the results, calculations with the finite element method were used (Solid Edge software).

Figure 3: FEM calculations

The results are consistent with sufficient accuracy.

V. CONCLUSIONS

The solution of large deflections by the numerical method has become effective. This solution can be used under various load conditions. FEM was used to verify the obtained results (Solid Edge software). As the obtained results show, the compliance of the results is satisfactory. This method can be used in theoretical problems.

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