A New Class of Nearly Open Sets In Topological Spaces

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Abstract

In this paper we introduce a new class of sets, namely semi-regular-open sets. We give a characterization of semi-regular*-open sets. We also define semi-regular*-interior of a subset. Further we study some fundamental properties of semi-regular*-open sets.*

Keywords: *semi-regular*-open set, semi-regular*-interior.*

I. INTRODUCTION

Norman Levine [3] introduced semi-open sets in topological spaces in 1963. Since the introduction of semi-open sets, many generalizations of various concepts in topology were made by considering semi-open sets instead of open sets.Levine [4] also defined and studied generalized closed sets in 1970. Dunham [2] introduced the concept of generalized closure using Levine's generalized closed sets and studied some of its properties. In 1937, regular open sets were introduced and used to define the semi-regularization space of a topological space. Quite recently the authors [8] introduced and studied some new concepts namely regular*-open sets.

In this paper, we introduce a new class of sets, namely semi-regular*-open sets, using the generalized closure operator Cl* due to Dunham. We further show that the concept of semi-regular*- open sets is weaker than the concept of regular*-open sets but stronger than the concept of semi-pre open sets. We investigate fundamental properties of semi-regular*-open sets. We also define semi-regular*-interior of a subset and study some of its basic properties.

II. PRELIMINARIES

Throughout this paper (X, τ) will always denote a topological space on which no separation axioms are assumed, unless explicitly stated. If A is a subset of a space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively

Definition 2.1: A subset A of a space X is **generalized closed** (briefly g-closed) [4] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X

Definition 2.2: If A is a subset of a space X, the **generalized closure** [2] of A is defined as the intersection of all g-closed sets in X containing A and is denoted by $Cl^*(A)$.

Definition 2.3: A subset A of a topological space (X, τ) is **semi-open** [3] (respectively **semi*-open** [11]) if there is an open set U in X such that $U \subseteq A \subseteq Cl(U)$ (respectively $U \subseteq A \subseteq Cl^*(U)$) or equivalently if $A \subseteq Cl(Int(A))$ (respectively $A \subseteq Cl^*(Int(A))$).

Definition 2.4: A subset A of a topological space (X, τ) is **pre-open** [5] if $A \subseteq Int(Cl(A))$

Definition 2.5: A subset A of a topological space (X, τ) is **α-open** [7] if $A \subseteq Int(Cl(Int(A)))$

Definition 2.6: A subset A of a topological space (X, τ) is **semi-preopen** [1] = β – open if $A \subseteq Cl(Int(Cl(A)))$

Definition 2.7: A subset A of a topological space (X, τ) is **regular-open** [6] if $A = Int(Cl(A))$.

Definition 2.8: A subset A of a topological space (X, τ) is said to be **regular*-open** [8] if A = Int(Cl*(A)).

Definition 2.9: The ***δ-interior*** [12] of A is defined as the union of all regular-open sets of X contained in A. It is denoted by **δ*Int*(*A*).**

Definition 2.10: A subset A of a topological space (X, τ) is **δ-open** [10] if $A = \delta Int(A)$.

Definition 2.11: A subset A is **semi*-δ -open** [9] if $A \subseteq Cl^*(\delta Int(A))$.

The class of all semi-open (respectively semi*-open , pre-open, α-open, semi-preopen, regular-open, regular*-open, δ-open and δ-semi-open) sets in (X, τ) is denoted by SO(X) (respectively S*O(X), PO(X), αO(X), SPO(X), RO(X), R*O(X), δ O(X) and S*δO(X)

Definition 2.12: The semi-interior[3] (respectively semi*-interior [12] , pre-interior [6], α-interior[7], semi-pre-interior[1], δ-interior[12] and semi* δ-interior[9]) of a subset A is defined to be the union of all semi-open (respectively semi*-open , pre-open, α-open, semi-preopen, regular-open, regular*-open and semi* δ-open) subsets of A. It is denoted by $sInt(A)$ (respectively $s^*Int(A)$, $pInt(A)$, $\alpha Int(A)$, $spInt(A)$, $\delta Int(A)$and $s^*\delta Int(A)$).

Definition 2.13: A topological space X is $T_{1/2}$[4] if every g-closed set in X is closed.

Definition 2.14: [13] A space X is locally indiscrete if every open set in X is closed.

Remark 2.15: [11] Every open set is semi*-open.

Remark 2.16: [8] Every regular*-open set is open.

Remark 2.17: [6] Every regular open set is open.

Remark 2.18: [11] Every α-open set is semi-open.

Remark 2.19: [10] Every δ-0pen set is open.

Remark 2.20: [9] Every semi*δ-open is semi*-open

Theorem 2.21: [2] Cl^* is a Kuratowski closure operator in X.

III. SEMI-REGULAR*-OPEN

Definition 3.1: A subset A of a topological space(X,τ) is called a **semi-regular*-open set** if there is a regular*-open set U in X such that $U \subseteq A \subseteq Cl(U)$.

Notation: The class of all semi- regular* open sets in (X,τ) is denoted by SR*O(X,τ)or simply SR*O(X).

Theorem 3.2: A subset A of X is semi-regular*-open if $A \subseteq Cl(Int(Cl^*(A)))$.

Proof: Assume A is semi- regular*- open set then there exist a regular*-open set U in X such that $U \subseteq A \subseteq Cl(U)$. Now $U \subseteq A \Rightarrow U = Int(Cl^*(U)) \subseteq Int(Cl^*(A)) \Rightarrow Cl(U) \subseteq Cl(Int(Cl^*(A))) \Rightarrow A \subseteq Cl(U) \subseteq Cl(Int(Cl^*(A))) \Rightarrow A \subseteq Cl(Int(Cl^*(A)))$

Remarks 3.3: In any topological space (X,τ), $\phi$and X are semi-regular*-open sets.

Theorem 3.4: If $\{A_\alpha\}$ is a collection of semi-regular*-open sets in X, then $\cup A_\alpha$ is also semi-regular*-open in X.

Proof: Let $\{A_\alpha\}$ be a collection of semi-regular*-open sets in X .Since each A_α is semi-regular*-open, $A_\alpha \subseteq Cl(Int(Cl^*(A_\alpha)))$. This implies $\cup A_\alpha \subseteq \cup (Cl(Int(Cl^*(A_\alpha)))) \subseteq (Cl(\cup Int(Cl^*(A_\alpha)))) \subseteq (Cl(Int(\cup Cl^*(A_\alpha)))) \subseteq (Cl(Int(Cl^*(\cup A_\alpha))))$. Hence $\cup A_\alpha$ is also semi-regular*-open in X.

Remark 3.5: The intersection of two semi-regular*-open sets need not to be semi-regular*-open as seen from the following example.

Example 3.6: Let X = $\{a, b, c\}$ and τ = $\{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. In the space$(X, \tau)$, the subsets A =$\{a, b\}$ and B = $\{b, c\}$ are semi-regular*-open but $A \cap B = \{b\}$ is not semi-regular*open.

Theorem 3.7: If A is semi-regular*-open in X and B is open in X, then $A \cap B$ is semi-regular*-open.

Proof: Since A is semi-regular*-open in X, there is a regular*-open set U such that U⊆A⊆Cl(U) since B is open, we have $U \cap B \subseteq A \cap B \subseteq Cl(U) \cap B \subseteq Cl(U \cap B)$. Since $U \cap B$ is regular*-open, by definition 3.1 $A \cap B$ is semi-regular*-open in X.

Theorem 3.8: S R*O(X, τ) forms a topology on X if and only if it is semi-regular*-open under finite intersection.

Proof: Follows from remarks 3.3 and theorem 3.4.

Theorem 3.9:

(i)Every semi*-open set is semi-regular*-open.

(ii) Every semi-open set is semi-regular*-open

(iii)Every open set is semi-regular*-open

(iv)Every regular*-open set is semi-regular*-open

(v)Every regular-open set is semi-regular*-open

(vi)Everyα-open set is semi-regular*-open

(vii)Every δ-open set is semi-regular*-open

(viii)Every semi*δ-open is semi-regular*-open

Proof: (i)Let A be a semi*– open set. Then A⊆Cl*$(Int(A))$⊆Cl$(Int(A))$⊆ Cl $(Int(Cl^*(A)))$

Hence A is semi- regular*-open.

(ii) Let A be a semi-open set. Then A⊆Cl$(Int(A))$⊆Cl$(Int(Cl^*(A)))$.

(iii) Follows from remark 2.15 and theorem 3.9(i)

(iv) Follows from remark 2.16 and theorem 3.9(iii)

(v) Follows from remark 2.17 and theorem 3.9(iii)

(vi) Follows from remark 2.18 and theorem 3.9(ii)

(vii) Follows from remark 2.19 and theorem 3.9(iii).

(viii) Follows from remark 2.20 and theorem 3.9(i)

Remark 3.10: The converse of each of the statements in Theorem 3.9 is not true as shown in the following examples:

Example 3.11: In the space X = $\{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.Here the set $\{b, c, d\}, \{a, c, d\}$is semi-regular*-open but not semi*-open

Example3.12: In the space X = $\{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\},$

$\{a, c, d\}\}$. Here the set $\{a, b, d\}$ is semi-regular*-open but not semi-open.

Example 3.13: In the space X = $\{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$.Here the set $\{a, b\}, \{b, c\}$ are semi-regular*-open but not open and $\{a, b\}, \{b, c\}, \{a, c\}$ are semi-regular*-open but not regular*-open.

Example 3.14: In the space X = $\{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, d\}, \{a, c, d\}\}$. Here the set $\{a\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}$ is semi-regular*-open but not regular-open and $\{a,b\},\{b,c\},\{a,b,d\}$ is semi-regular*-open but not α-open.

Example 3.15: In the space X = $\{a,b,c,d\}$ and $\tau = \{X, \phi, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}\}$. Here the set $\{a,b\},\{c,d\},\{a,c\},\{b,d\},\{a,b,c\},\{b,c,d\},\{a,c,d\}$, $\{a,b,d\}$ is semi-regular*-open but not δ-open and $\{a,b\},\{a,c\},\{a,b,c\},\{b,c,d\},\{a,c,d\},\{a,b,d\}$ but not semi* δ-open.

Theorem 3.16: Every semi-regular*-open set is semi -pre-open.

Proof: Let A be semi-regular*-open then A$\subseteq$ $Cl\big(Int(Cl^*(A))\big) \subseteq Cl\big(Int(Cl(A))\big)$. Hence A is semi-preopen.

Remark 3.17: The converse of above Theorem is not true as shown in the following example:

Example 3.18: In the space X = $\{a,b,c,d\}$ and $\tau = \{X, \phi, \{a\}, \{b,c,d\}\}$. Here the set $\{b\},\{c\},\{d\},\{a,b\},\{b,c\},\{c,d\},\{a,c\},\{a,d\},\{b,d\},\{a,b,c\},\{a,c,d\},\{a,b,d\}$ is semi-pre-open but not semi-regular*-open.

Theorem 3.19: In any topological space (X, τ), R*O(X, τ)⊆ SR*O(X, τ)⊆SPO(X, τ) .That is the class of semi-regular*-open set is placed between the class of regular*-open sets and the class of semi-pre-open sets.

Proof: Follows from Theorem 3.9 and Theorem 3.16

Remark 3.20: (i) If (X, τ) is a locally indiscrete space, τ = δO(X, τ) = S*δO(X, τ) = SR*O(X, τ) = S*O(X, τ)= SO(X, τ) = αO(X, τ) =RO(X, τ).=R*O(X, τ)

(ii) The inclusions in Theorem 3.19 may be strict and equality may also hold. This can be seen from the following examples.

Example 3.21: In the topological space (X, τ) where $X = \{a, b, c, d\}$ and τ ={X, ϕ, {a}, {b}, {a, b}, {a, b, c}}, R*O(X, τ) ⊊ SR*O(X, τ) = SPO(X, τ).

Example 3.22: In the topological space (X, τ) where $X = \{a, b, c, d\}$ and τ ={X, ϕ, {a}, {b}, {a, b}, {c, d}, {b, c, d}, {a, c, d}}, R*O(X,τ) ⊊SR*O(X,τ) ⊊SPO(X, τ)

Example 3.23: In the topological space (X, τ) where $X = \{a, b, c, d\}$ and τ ={X, ϕ, {a}, {b, c, d}},

R*O(X, τ) = SR*O(X, τ) ⊊ SPO(X, τ).

Remark 3.24: The concept of semi-regular*-open sets and pre-open sets are independent as seen from the following examples:

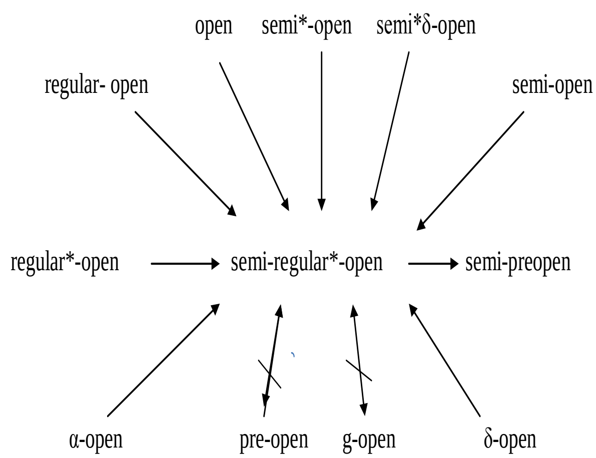
Example 3.25: In the topological space (X, τ) where $X = \{a, b, c, d\}$ and τ ={X, ϕ, {a}, {b}, {a, b}, {c, d}, {b, c, d}, {a, c, d}}, the subsets $\{c\},\{d\},\{b,c\},\{a,c\},\{a,d\},\{b,d\},\{a,b,c\}$ is pre-open but not semi-regular*-open.

Example 3.26: In the topological space (X, τ) where $X = \{a, b, c, d\}$ and τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}}, the subsets $\{b,c\},\{a,c\},\{a,d\},\{b,d\},\{b,c,d\},\{a,c,d\}$ is semi-regular*-open but not pre-open.

Remark 3.27: The concept of semi-regular*-open sets and g-open sets are independent as seen from the following examples:

Example 3.28: In the topological space (X, τ) where X = {a, b, c, d} and τ = {X, ϕ, {a}, {b}, {a, b}, {a, b, c}}, the subsets {d} is g-open but not semi-regular*-open.

Example 3.29: In the topological space (X, τ) where X = {a, b, c, d} and τ = {X, ϕ, {a}, {b}, {c}, {a, b}, {b, c}, {a, c},{a ,b ,c}}, the subsets {a, d} is semi-regular*-open but not g-open.



IV. SEMI-REGULAR*-INTERIOR OF A SET

Definition 4.1: The **semi-regular*-interior** of A is defined as the union of all semi- regular* open sets of X contained in A. It is denoted by sr*(Int (A)).

Theorem 4.2: If A is any subset of a topological space (X, τ), then

(i) *sr*Int*(A) is the largest semi-regular*-open set contained in A.

(ii) A is semi-regular*-open if and only if *sr*Int*(A) $=A$.

Proof: (i) Being the union of all semi-regular*-open subsets of A, by Theorem 3.4, *sr*Int*(A) is semi-regular*-open and contains every semi-regular*-open subset of A. This proves (i).

(ii) A is semi-regular*-open implies *sr*Int*(A) $=A$ is obvious from Definition 4.1. On the other hand, suppose *sr*Int*(A) $=A$. By (i), *s*δInt*(A) is semi-regular*-open and hence A is semi-regular*-open.

Theorem 4.3: (Properties of Semi-Regular*-Interior)

In any topological space (X, τ), the following statements hold:

(i) $sr^*Int(\phi)= \phi$

(ii) $sr^*Int(X)=X$

If A and B are subsets of X,

(iii) $sr^*Int(A) \subseteq A$

(iv) $A\subseteq B \Rightarrow sr^*Int(A) \subseteq sr^*Int(B)$

(v) $sr^*Int(sr^*Int(A)) = sr^*Int(A)$

(vi) $r^*(Int (A)) \subseteq sr^*(Int(A)) \subseteq sp(Int(A))$

(vii) $sr^*Int(A\cup B) \supseteq sr^*Int(A) \cup sr^*Int(B)$

(viii) $sr^*Int(A\cap B) \subseteq sr^*Int(A) \cap sr^*Int(B)$

Proof: (i), (ii), (iii) and (iv) follow from Definition 4.1. By Theorem 4.2(i),sr* Int(A) is semi-regular*-open and by Theorem 4.2(ii),sr*Int(sr*Int(A)) = sr*Int(A). Thus (v) is proved. The statements (vi) follows from Theorem 3.9(iv) and theorem 3.16. Since $A \subseteq A\cup B$, from statement (iv) we have $sr^*Int(A) \subseteq sr^*Int(A\cup B)$ Similarly, $sr^*Int(B) \subseteq sr^*Int(A\cup B)$.This proves (vii). The proof for (viii) is similar.

Remark 4.4: In (vi) of Theorem 4.3, each of the inclusions may be strict and equality may also hold. This can be seen from the following examples:

Example 4.5: In the space (X, τ) where $X=\{a, b, c, d\}$ and $\tau = \{X,\phi,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\}\}$ Let $A=\{a,b\}$ Then $r^*(Int(A))=sr^*(Int(A))=sp(Int(A))=\{a,b\}=A$

Let $B=\{a,b,d\}$ Then $r^*(Int(B))=\{a,b\}$; $sr^*(Int(B))=sp(Int(B))=\{a,b,d\}$

Here $r^*(Int(B))\subsetneq sr^*(Int(B))=sp(Int(B))=B$

Let $C=\{c, d\}$ Then $r^*(Int(A))=sr^*(Int(A))=sp(Int(A))=\phi$

Here $r^*(Int(A))=sr^*(Int(A))=sp(Int(A))\subsetneq C$

Remark 4.6: The inclusions in (vii) and (viii) of Theorem 4.3 may be strict and equality may also hold. This can be seen from the following examples.

Example 4.7: Consider the space (X, τ) in Example 4.5

Let $A= \{a, b\}$ and $B = \{b, d\}$ Then $A\cup B = \{a, b, d\}$ $sr^*Int(A) = \{a, b\}$; $sr^*Int(B) = \{b, d\}$.Then $sr^*Int(A\cup B)=\{a,b,d\}$

Here $sr^*Int(A\cup B)= sr^*Int(A)\cup sr^*Int(B)$

Let $C = \{a, d\}$ and $D=\{a, c\}$ Then $C\cap D=\{a\}$

$sr^*Int(C)=\{a, d\}$; $sr^*Int(D)=\{a\}$ Then $sr^*Int(C\cap D)=\{a\}$

Here $sr^*Int(A\cap B) = sr^*Int(A)\cap sr^*Int(B)$

Let E ={a, d} and F = {b, d} Then E∩ F={d}

sr*Int(E) = {a, d} ; sr*Int(F) = {b, d} ; sr*Int(E∩F) = ϕ;

sr*Int(E) ∩ sr*Int(F) ={d}

Here sr*Int(E∩F) ⊊ sr*Int(E) ∩ sr*Int(F)

Let G={a, b} and H={c, d} Then G∪H={a, b, c, d}

sr*Int(G)={a, b}; sr*Int(H)={ϕ}; sr*Int(G∪H}={a, b, c, d}; sr*Int(G) ∪sr*Int(H)={a, b}

Here sr*Int(G) ∪sr*Int(H)⊊sr*Int(G∪H}

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