Two Phase Service Repairable M^X/G/1 Queueing Models with Finite Number of Immediate Feedbacks

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Abstract-*In queueing situations, it is quite significant to allow every unsatisfied customers to demand for re services (feedback) before leaving the system. But to avoid the complexity, most of the existing research articles examine queueing systems under infinite number of feedback services which is offensive. Earlier Kalidass and Kasturi (2013) analysed steady results of M/G/1 queueing model with finite number of feedbacks for a reliable server.The present paper analyses an unreliable* $M^X/G, G_{i\,(1\leq i\leq C)}/1$ *queueing system with two phases of heterogeneous service in which the second phase consists of multi-optional heterogeneous service facilities. The arriving customers is said to complete the first round of service, if they undergo the first phase service and one of the second optional services. After completing each service, the unsatisfied customers may demand for re-services finitely many times before leaving the system. The server is subject to unpredictable breakdowns during busy period and the service interrupted customers stay in the service facility and complete the remaining service as soon as the server is fixed. The present paper generalizes several results of feedback queueing models including that of Kalidass and Kasturi (2013).*

Keywords: *Two Phase service, Immediate Bernoulli feedback, Unpredictable Breakdowns, Performance Measures.*

1. INTRODUCTION

Many queueing situations have the feature that the customers may be served repeatedly for certain reasons. When the service of a customer is unsuccessful, it may be retried again and again until a successful service completion occurs. The concept of feedback was introduced by Takacs (1963) and since then many papers have appeared about this topic. He considered an M/G/1 Bernoulli feedback queue with single class customers and obtained the distributions of queue size and the total response time of a customer. Disney and Konig (1985) have given an overview of the literature concerning Bernoulli feedback studies. Fewer results are known for feedback queueing systems in which the feedback policy is not Bernoulli. Baskett et al.(1975) obtained the product form of the joint queue size distribution for the M/M/1 queueing system with several types of customers and general feedback policy. Choi and Tae-Sung (2003), Choudhury and Paul (2005) ,BadamchiZadeh and Shahkar (2008) and Thangaraj and Vanitha (2010) derived the queue size distribution for M/G/1 queue with two phases of heterogeneous services and Bernoulli feedback system. Most of the literature relating to feedback queueing models assume that the customers , if unsatisfied with the service will join the tail of the queue to claim for the next round service and such demand may be repeated infinitely many times before leaving the system. But it is not practical to allow the customers to feedback infinitely many times. In the present chapter a general bulk arrival queueing model with finite number of immediate feedbacks is considered. The server is subjected to unpredictable breakdowns, operates two phases of heterogeneous services. The first phase consists of single essential service and multi optional service facilities exist in the second phase. A customer is said to complete a first round service if he undergoes the first phase service and any one of the second phase services. After having completed the first round service, the customer is permitted to demand for immediate reservice.

2. MODEL DESCRIPTION

The system has the following specifications.

A. Arrival Pattern

Customers arrive in batches in accordance with a time homogeneous Poisson process with random batch size X, group arrival rate λ and probability distribution $g_k = \Pr(X = k), k = 1,2,3,\ldots$ (i.e.,) the probability that a batch of k units arrive in an infinitesimal interval (t, t+h) is $\lambda g_k h + o(h)$. The customers who arrive in batches join the system and form a single waiting line based on the order of the batches. The customers within a batch are pre-ordered. There is a single server who serves the customers one by one according to the order in the queue during busy period. i.e., First Come First Served (FCFS) queue discipline is followed.

B. BUSY PERIOD AND BREAKDOWN PERIOD

During busy period, the server provides two phases of heterogeneous service in succession to each customer. A single essential service is provided in the first phase and multi-optional service facilities in second phase. Every arriving customer after receiving essential service in the first phase will opt for a certain $i^{th} (1 \le i \le C)$ optional service with probability r_i where $\sum_{i=1}^{C} r_i = 1$ from the second phase.The second phase service immediately commences after completing the first phase service. All the services are provided by the same server. The service times in the first and second phases of service are independently distributed random variables. The distribution function, Laplace-Stieltjes transform of the service time in the first phase are respectively denoted by S(t), $S^*(\theta)$ with finite first and second moments.

The service time of the $i^{th} (1 \le i \le C)$ optional services of the second phase has a distribution function $S_i(t) (1 \le i \le C)$ with Laplace- Stieltjes transform $S_i(\theta)$. The first and second moments of the distribution are finite and respectively given by $E(S_i) = -S_i^{*'}(0)$ and $E(S_i^2) = S_i^{*''}(0)$.

The service time of the first round of service of a customer is thus S+S_i (for some i=1 to C) and it is termed as primary round or fresh service.

The customer who finishes the first round of service either feeds back immediately into the system and starts the first phase of service with a probability f_1 or leaves the system forever with probability $1 - f_1$. After the completion of the feedback service, the customer may again go in for a third round of service by entering the phase 1 service with a probability $f_2 < f_1$ or the customer may depart from the system with a probability $1 - f_2$. The feedback process will continue until either the customer is satisfied or demands m rounds of services, after which the customer has to leave the system. The next customer in the queue can go into the system only after successful completion of all the feedback rounds of the performing the first phase service of a customer or busy performing one of the services in the second phase.

The server is subject to unpredictable breakdowns while serving the customers. The breakdowns occur according to the Poisson process with rate **a** in first phase and at the rate $\mathbf{a}_2^{(i,j)}$ during the i^{th} type of j^{th} feedback service in the second phase. As soon as a breakdown occurs, the server is sent for repair immediately and the customer whose service is interrupted stays in the service facility to complete the remaining service. The repair times R_1, $R_2^{(i,j)}$ $1 \le i \le C, 0 \le j \le m-1$ are arbitrarily distributed with probability distribution functions $R_1(t)$ and $R_2^{(i,j)}(t)$ according as the interruption occurs in phase 1 or phase 2 in the i^{th} type of service due to the j^{th} feedback. Immediately after the server is fixed, he starts to serve the customer, whose service is interrupted and the service time is assumed to be cumulative.

Let

$$C(t)=\begin{cases} 0, \text{the server is idle,} \\ 1, \text{the server is performing the first essential service,} \\ 2, \text{the server is performing the second phase service,} \\ 3, \text{if the server is under repair in first phase,} \\ 4, \text{if the server is under repair in the } i^{th} \text{ type} \\ \quad j^{th} \text{ feedback service of second phase} \end{cases}$$

Let $N_s(t)$ be the number of customers in the system at time t. Let X(t) denote the remaining service time of a customer receiving service in the first phase and $Y_i(t)$ denote the remaining service time of a customer receiving $i^{th}(1 \le i \le C)$ optional service in the second phase of service at time t.

Thus the state space $\{N_s(t), \delta(t)\}$ where $\delta(t) = \left(0, S^0(t),\ S_i^0(t),\ R_1^0(t),\ {R_2^{(I,j)}}^0(t),\right)$

according as Y(t)=0,1,2,3 and 4 respectively follows bivariate Markov process.

The following joint probability functions are defined at time t, for further analysis:

$PI(t) = \Pr\{N_s(t) = 0, Y(t) = 0\}$, when the server is idle.

For $n \ge 0$,

$P_{1,n}^0(x,t)dt = \Pr\{N_s(t) = n, x < S^0(t) \le x + dt, Y(t) = 1\}$, a customer is being served in first phase primary service.

For $0 \le j \le m-1$ and $n \ge 1$,

$P_{1,n}^j(x,t)dt = \Pr\{N_s(t) = n, x < S^0(t) \le x + dt, Y(t) = 1\}$, a customer is being served in first phase of the j^{th} feedback.

For $1 \le i \le C, 0 \le j \le m-1$

$P_{2,n}^{(i,j)}(x,t)dt = \Pr\{N_s(t) = n, x < S_i^0(t) \le x + dt, Y(t) = 2\}$, a customer is being served in the i^{th} optional service of the second phase during j^{th} feedback.

j=0 corresponds to fresh or primary i^{th} second phase service.

$BR_{1,n}(x,y,t)dt = \Pr\{N_s(t) = n, S^0(t) = x, y < R_1^0(t) \le y + dt, Y(t) = 3\}$, a customer is waiting for first phase service due to breakdown.

$BR_{2,n}^{(i,j)}(x,y,t)dt = \Pr\left\{N_s(t) = n, S_i^0(t) = x, y < {R_2^{(i,j)}}^0(t) \le y + dt, Y(t) = 4\right\}$, a customer is waiting for the i^{th} optional service of the second phase during the j^{th} feedback due to breakdown.

Further,

$P_{1,n}^0(0), P_{1,n}^j(0), P_{2,n}^{(i,j)}(0), BR_{1,n}(0), BR_{2,n}^{(i,j)}(0)$ denote the probability that there are n customers in the system at the termination of service time and repair time.

Assuming that at steady state, the probabilities are independent of time t, we have

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_{1,n}^0(x,t) = \frac{d}{dx} P_{1,n}^0(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_{1,n}^j(x,t) = \frac{d}{dx} P_{1,n}^j(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_{2,n}^{(i,j)}(x,t) = \frac{d}{dx} P_{2,n}^{(i,j)}(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial y} BR_{1,n}(x,y,t) = \frac{d}{dx} BR_{1,n}(x,y),$$

$$\lim_{t\to\infty} \frac{\partial}{\partial y} BR_{2,n}^{(i,j)}(x,y,t) = \frac{d}{dx} BR_{2,n}^{(i,j)}(x,y)$$

$$\lim_{t\to\infty} \left(\frac{\partial}{\partial t} P_{1,n}^0(x,t) = \frac{\partial}{\partial t} P_{1,n}^j(x,t) = \frac{\partial}{\partial t} P_{2,n}^{(i,j)}(x,t)\right) = 0$$

$$\lim_{t\to\infty} \left(\frac{\partial}{\partial t} BR_{1,n}(x,y,t) = \frac{\partial}{\partial t} BR_{2,n}^{(i,j)}(x,y,t)\right) = 0$$

$$\lim_{t\to\infty} PI(t) = PI$$

$$\lim_{t\to\infty} P_{1,n}^0(x,t) = P_{1,n}^0(x)$$

$$\lim_{t\to\infty} P_{1,n}^j(x,t) = P_{1,n}^j(x)$$

$$\lim_{t\to\infty} P_{2,n}^{(i,j)}(x,t) = P_{2,n}^{(i,j)}(x)$$

$$\lim_{t\to\infty} BR_{1,n}(x,y,t) = BR_{1,n}(x,y)$$

$$\lim_{t\to\infty} BR_{2,n}^{(i,j)}(x,y,t) = BR_{2,n}^{(i,j)}(x,y)$$

C. System Size Distribution at Random Epoch

Observing the changes of states in the interval $(t, t+\Delta t)$ at any time t, the steady state equations are given by:

Idle State:

$$\lambda PI = \sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{2,1}^{(i,j)}(0)\left(1 - f_{j+1}\right)$$

Busy with First Phase (Primary Service):

$$-\frac{d}{dx}P_{1,n}^{0}(x)$$
$$= -(\lambda + a_1^0)P_{1,n}^{0}(x)$$
$$+\lambda\left(1-\delta_{1,n}\right)\sum_{k=1}^{n-1} P_{1,n-k}^{0}(x)\, g_k + PI\lambda g_n s(x)$$
$$+BR_{1,n}^{0}(x,0) + s(x)\sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{2,n+1}^{(i,j)}(0)\left(1 - f_{j+1}\right), n \ge 1$$

Busy with First Phase (Feedback Service):

$$\frac{d}{dx}P_{1,n}^{j}(x)$$
$$= -\left(\lambda + a_1^{(j)}\right)P_{1,n}^{j}(x)$$
$$+\lambda\left(1-\delta_{1,n}\right)\sum_{k=1}^{n-1} P_{2,n-k}^{j}(x)\, g_k + BR_{1,n}^{j}(x,0)$$
$$+\sum_{i=1}^{C} P_{2,n}^{(i,j-1)}(0)\, f_j s(x), 1 \le j \le m-1, n \ge 1$$

Busy with Second Phase

$$\frac{d}{dx}P_{2,n}^{(i,j)}(x)$$
$$= -\left(\lambda + a_2^{(i,j)}\right)P_{2,n}^{(i,j)}(x)$$
$$+\lambda\left(1-\delta_{1,n}\right)\sum_{k=1}^{n-1} P_{2,n-k}^{(i,j)}(x)\, g_k + BR_{2,n}^{(i,j)}(x,0)$$
$$+{P_{1,n}}^{j}(0) r_i s_i(x), 0 \le j \le m-1 n \ge 1$$

Breakdown in First Phase:

$$-\frac{\partial}{\partial y}BR_{1,n}^{j}(x,y)$$
$$= -\lambda BR_{1,n}^{j}(x,y)$$
$$+\lambda\left(1-\delta_{1,n}\right)\sum_{k=1}^{n-1} BR_{1,n}^{j}(x,y)\, g_k + a_1^{j} P_{1,n}^{j}(x) r_1^{j}(y)$$
$$n \ge 1, 0 \le j \le m-1$$

Breakdown in Second Phase:

$$-\frac{\partial}{\partial y}BR_{2,n}^{(i,j)}(x,y)$$
$$= -\lambda BR_{2,n}^{(i,j)}(x,y)$$
$$+\lambda\left(1-\delta_{1,n}\right)\sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)}(x,y)\, g_k$$
$$+a_2^{(i,j)} r_2^{(i,j)}(y) P_{2,n}^{(i,j)}(x)$$
$$n \ge 1, 1 \le i \le C, 0 \le j \le m-1$$

Laplace Stieltjes Transform (LST):

Let X be a non-negative random variable with distribution function F(.), then the LST $F^*(\theta)$ of the distribution is defined by

$$F^*(\theta) = \int_{x=0}^{\infty} e^{-\theta x}\, dF(t) \text{and } F^*(0) = 1.$$

The n^{th} moment of X, when it exists is given by

$$E(X^n) = \left[(-1)^n \frac{d^n}{d\theta^n} F^*(\theta)\right]_{\theta=0}.$$

Then the LST of the steady state equations are given by

$$\theta P_{1,n}^{0^*}(\theta) - P_{1,n}^{0}(0)$$

$$= (\lambda + a_1^0)P_{1,n}^{0^*}(\theta) - \lambda(1-\delta_{1,n})\sum_{k=1}^{n-1} P_{1,n-k}^{0^*}(\theta)\, g_k$$

$$-S^*(\theta)\lambda PIg_n - BR_{1,n}^{0^*}(\theta,0)$$

$$-S^*(\theta)\sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{2,n+1}^{(i,j)}(0)\left(1-f_{j+1}\right) \quad (1)$$

$$\theta P_{1,n}^{j^*}(\theta) - P_{1,n}^{j}(0)$$

$$= \left(\lambda + a_1^j\right)P_{1,n}^{j^*}(\theta) - \lambda(1-\delta_{1,n})\sum_{k=1}^{n-1} P_{1,n-k}^{j^*}(\theta)\, g_k$$

$$-BR_{1,n}^{j^*}(\theta,0) - \sum_{i=1}^{C} P_{2,n}^{(i,j-1)}(0)\, f_j S^*(\theta) \quad (2)$$

$$\theta P_{2,n}^{(i,j)^*}(\theta) - P_{2,n}^{(i,j)}(0)$$

$$= \left(\lambda + a_2^{(i,j)}\right)P_{2,n}^{(i,j)^*}(\theta)$$

$$-\lambda(1-\delta_{1,n})\sum_{k=1}^{n-1} P_{2,n-k}^{(i,j)^*}(\theta)\, g_k - P_{1,n}^{j}(0) r_i S_i^*(\theta)$$

$$-BR_{2,n}^{(i,j)^*}(\theta,0), 1 \le i \le C, 0 \le j \le m-1,$$

$$n \ge 1 \quad (3)$$

$$\theta_1 BR_{1,n}^{j^{**1}}(\theta,\theta_1) - BR_{1,n}^{j^*}(\theta,0)$$

$$= \lambda BR_{1,n}^{j^{**1}}(\theta,\theta_1) - a_1^j P_{1,n}^{j^*}(\theta) R_1^{j^{**1}}(\theta_1),$$

$$-\lambda(1-\delta_{1,n})\sum_{k=1}^{n-1} BR_{1,n-k}^{j^{**1}}(\theta,\theta_1)\, g_k$$

$$n \ge 1 \quad (4)$$

$$\theta_1 BR_{2,n}^{(i,j)^{**1}}(\theta,\theta_1) - BR_{1,n}^{(i,j)^*}(\theta,0)$$

$$= \lambda BR_{1,n}^{(i,j)^{**1}}(\theta,\theta_1)$$

$$-\lambda(1-\delta_{1,n})\sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)^{**1}}(\theta,\theta_1)\, g_k$$

$$-a_2^{(i,j)} P_{2,n}^{(i,j)^*}(\theta) R_2^{(i,j)^{*1}}(\theta_1)$$

$$n \ge 1, 1 \le i \le C, 0 \le j \le m-1 \quad (5)$$

$$\lambda PI = \sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{2,1}^{(i,j)}(0)\left(1-f_{j+1}\right) \quad (6)$$

D. *Probability Generating Functions:*

The following partial PGFs are introduced to analyse the model:

$$P_1^{0*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{0*}(\theta) z^n$$

$$P_1^{0}(z,0) = \sum_{n=1}^{\infty} P_{1,n}^{0}(0)\, z^n$$

$$P_1^{j*}(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^{j*}(\theta) z^n$$

$$P_1^{j}(z,0) = \sum_{n=1}^{\infty} P_{1,n}^{j}(0)\, z^n \;, 0 \le j \le m-1$$

$$P_2^{(i,j)^*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{(i,j)^*}(\theta) z^n$$

$$P_2^{(i,j)}(z,0) = \sum_{n=1}^{\infty} P_{2,n}^{(i,j)}(0) z^n, 1 \le i \le C, 0 \le j \le m-1$$

$$BR_1^{j^{**1}}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{1,n}^{j^{**1}}(\theta,\theta_1) z^n$$

$$BR_1^{j^*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}^{j^*}(\theta,0) z^n, 0 \le j \le m-1$$

$$BR_2^{(i,j)^{**1}}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,j)^{**1}}(\theta,\theta_1) z^n$$

$$BR_2^{(i,j)^*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,j)^*}(\theta,0)\, z^n$$

$$1 \le i \le C, 0 \le j \le m$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, partial generating functions are derived, through some algebraic manipulations.

The partial generating functions of the system size, when the server is in breakdown state during the first and second stage of service are obtained respectively by using equations (4) and (5) and are given by,

$$BR_1^{j*}(z,\theta,0) = a_1^j P_1^{j*}(z,\theta) R_1^{j*1}(w_X(z)) \tag{7}$$

$$BR_1^{j**1}(z,\theta,\theta_1) = \frac{a_1^j P_1^{j*}(z,\theta)\left(R_1^{j*1}(w_X(z)) - R_1^{j*1}(\theta_1)\right)}{\theta_1 - w_X(z)} \tag{8}$$

$$BR_2^{(i,j)^*}(z,\theta,0) = a_2^{(i,j)} P_2^{(i,j)^*}(z,\theta) R_2^{(i,j)^*1}(w_X(z)) \tag{9}$$

$$BR_2^{(i,j)^{**1}}(z,\theta,\theta_1) = \frac{a_2^{(i,j)} P_2^{(i,j)^*}(z,\theta)\left(R_2^{(i,j)^*1}(w_X(z)) - R_2^{(i,j)^*1}(\theta_1)\right)}{\theta_1 - w_X(z)}$$

$$1 \le i \le C, 0 \le j \le m-1 \tag{10}$$

Multiplying (3) by z^n , we get,

$$\theta P_2^{(i,j)^*}(z,\theta) - P_2^{(i,j)}(z,0) = \left(\lambda + a_2^{(i,j)}\right) P_2^{(i,j)^*}(z,\theta) - \lambda X(z) P_2^{(i,j)^*}(z,\theta) - P_1^j(z,0) r_i S_i^*(\theta) - BR_2^{(i,j)^*}(z,\theta,0),$$

$$1 \le i \le C, 0 \le j \le m-1 \tag{11}$$

Substituting for $BR_2^{(i,j)^*}(z,\theta,0)$ in (11),

$$[\theta - ha_2^{(i,j)}(w_X(z))] P_2^{(i,j)^*}(z,\theta) = P_2^{(i,j)}(z,0) - P_1^j(z,0) r_i S_i^*(\theta) \tag{12}$$

At $\theta = ha_2^{(i,j)}(w_X(z))$,

$$P_2^{(i,j)}(z,\theta) = P_1^j(z,0) r_i S_i^*\left(ha_2^{(i,j)}(w_X(z))\right)$$

$$\sum_{i=1}^{C} P_2^{(i,j)}(z,0) = \sum_{i=1}^{C} r_i S_i^*\left(ha_2^{(i,j)}(w_X(z))\right) P_1^j(z,0)$$

$$= k_j(z) P_1^j(z,0) \tag{13}$$

where

$$k_j(z) = \sum_{i=1}^{C} r_i S_i^*\left(ha_2^{(i,j)}(w_X(z))\right) \quad 1 \le j \le m-1 \tag{13.1}$$

Using this in (12),

$$P_2^{(i,j)^*}(z,\theta) = \frac{P_1^j(z,0) r_i \left[S_i^*\left(ha_2^{(i,j)}(w_X(z))\right) - S_i^*(\theta)\right]}{\left[\theta - ha_2^{(i,j)}(w_X(z))\right]} \tag{14}$$

At $\theta = 0$,

$$P_2^{(i,j)^*}(z,0) = \frac{P_1^j(z,0) r_i \left[1 - S_i^*\left(ha_2^{(i,j)}(w_X(z))\right)\right]}{ha_2^{(i,j)}(w_X(z))} \tag{15}$$

The PGF of the system size when the server is busy with first stage service at the j[th] feedback is obtained by using the equation (2),

$$[\theta - ha_1^j(w_X(z))] P_1^{j^*}(z,\theta) = P_1^j(z,0) - \sum_{i=1}^{C} P_2^{(i,j-1)}(0) f_j S^*(\theta),$$

$$1 \le j \le m-1 \tag{16}$$

At $\theta = ha_1^j(w_X(z))$,

$$P_1^j(z,0) = \sum_{i=1}^{C} P_2^{(i,j-1)}(0) f_j S^*\left(ha_1^j(w_X(z))\right),$$

$$0 \le j \le m-1 \tag{17}$$

Using (13),

$$P_1^j(z,0) = k_{j-1}(z) f_j S^*\left(ha_1^j(w_X(z))\right) P_1^{j-1}(z,0),$$

$$0 \le j \le m-1.$$

On recursion,

$$P_1^j(z,0) = \prod_{r=0}^{j-1} k_r(z) \prod_{s=1}^{j} f_s\, S^*\left(ha_1^s\left(w_X(z)\right)\right) P_1^0(z,0)\,,$$

$$1 \le j \le m-1 \qquad (18)$$

Substituting for $P_1^j(z,0)$ in (16), then at $\theta = 0$,

$$P_1^{j^*}(z,0) = \prod_{r=0}^{j-1} k_r(z) \prod_{l=1}^{j} f_l \prod_{s=1}^{j-1} S^*\left(ha_1^s\left(w_X(z)\right)\right) \frac{1-S^*\left(ha_1^j\left(w_X(z)\right)\right)}{ha_1^j\left(w_X(z)\right)} P_1^0(z,0) \qquad (19)$$

Substituting for $P_1^j(z,0)$ from equation (18) in (15),

$$\sum_{i=1}^{C} P_2^{(i,j)}(z,0) = \prod_{r=0}^{j-1} k_r(z) \prod_{s=1}^{j} f_s\, S^*\left(ha_1^s\left(w_X(z)\right)\right) P_1^0(z,0)\,,$$

$$0 \le j \le m-1 \qquad (20)$$

$$P_2^{(i,j)^*}(z,0) = \prod_{r=0}^{j-1} k_r(z) \prod_{s=1}^{j} f_s S^*\left(ha_1^s\left(w_X(z)\right)\right) \frac{r_i\left[1-S_i^*\left(ha_2^{(i,j)}\left(w_X(z)\right)\right)\right]}{ha_2^{(i,j)}\left(w_X(z)\right)} P_1^0(z,0) \qquad (21)$$

The PGF of the system size when the server is busy with first stage of service from equation (1),

$$\text{i.e.,} \left[\theta - ha_1^0\left(w_X(z)\right)\right] P_1^{0^*}(z,\theta)$$

$$= P_1^0(z,0) - \lambda X(z) PIS^*(\theta)$$

$$-\frac{1}{z}\sum_{j=0}^{m-1}\sum_{i=1}^{C} P_2^{(i,j)}(z,0)\left(1-f_{j+1}\right) S^*(\theta) \qquad (22)$$

Adding the equation (6), multiplying by $S^*(\theta)$ with the above equation,

$$\left[\theta - ha_1^0\left(w_X(z)\right)\right] P_1^{0^*}(z,\theta)$$

$$= P_1^0(z,0) + PIw_X(z)S^*(\theta)$$

$$-\sum_{j=0}^{m-1}\sum_{i=1}^{C} \frac{P_2^{(i,j)}(z,0)}{z}\left(1-f_{j+1}\right) S^*(\theta) \qquad (23)$$

Using equation (20),

$$\left[\theta - ha_1^0\left(w_X(z)\right)\right] P_1^{0^*}(z,\theta)$$

$$= \frac{P_1^0(z,0)}{z}\left[z - \sum_{j=0}^{m-1}\prod_{r=0}^{j} k_r(z)\, f_r S^*\left(ha_1^r\left(w_X(z)\right)\right)\left(1 - f_{j+1}\right) S^*(\theta)\right]$$

$$+PIw_X(z)S^*(\theta) \qquad (24)$$

At $\theta = ha_1^0\left(w_X(z)\right)$ we get,

$$P_1^0(z,0) = \frac{-zPIw_X(z)S^*\left(ha_1^0\left(w_X(z)\right)\right)}{z-\phi(z)} \qquad (25)$$

where

$$\phi(z) = \sum_{j=0}^{m-1}\left(1-f_{j+1}\right)\prod_{s=0}^{j} k_s(z) f_s \quad \text{and}$$

$$k_s(z) = \prod_{r=0}^{j} k_r(z) S^*\left(ha_1^r\left(w_X(z)\right)\right)$$

Substituting for $P_1^0(z,0)$ in (24), (19) and (21), $P_1^{j^*}(z,0), P_2^{(I,j)^*}(z,0)$ for $1 \le i \le C$, $0 \le j \le m-1$ are obtained.

Thus the partial generating functions of the system size are expressed in terms of the only unknown PI and are listed below.

$$P_1^{0^*}(z,0) = \frac{zPIw_X(z)\left[S^*\left(ha_1^0\left(w_X(z)\right)\right)-1\right]}{ha_1^0\left(w_X(z)\right)\left[z-\phi(z)\right]} \qquad (26)$$

For $1 \le j \le m-1$,

$$P_1^{j^*}(z,0) =$$

$$\frac{zPIw_X(z)}{z-\phi(z)}\prod_{r=0}^{j-1}k_r(z)\prod_{l=1}^{j}f_l\frac{\left[S^*\left(ha_1^j\left(w_X(z)\right)\right)-1\right]}{ha_1^j\left(w_X(z)\right)}P_1^0(z,0) \quad (27)$$

For $1 \le i \le C, 0 \le j \le m-1$,

$$P_2^{(i,j)^*}(z,0)$$

$$=\left(\frac{zPIw_X(z)}{z-\phi(z)}\prod_{r=0}^{j-1}k_r(z)\prod_{s=1}^{j}f_sS^*\left(ha_1^s\left(w_X(z)\right)\right)\right)$$

$$\frac{r_i\left[S_i^*\left(ha_2^{(i,j)}\left(w_X(z)\right)\right)-1\right]}{ha_2^{(i,j)}\left(w_X(z)\right)} \quad (28)$$

$$BR_1^{j^{**1}}(z,0,0)$$

$$=\frac{a_1^jP_1^{j^*}(z,0)\left(1-R_1^{j^{*1}}\left(w_X(z)\right)\right)}{w_X(z)} \quad (29)$$

$$BR_2^{(i,j)^{**1}}(z,0,0)$$

$$=\frac{a_2^{(i,j)}P_2^{(i,j)^*}(z,0)\left(1-R_2^{(i,j)^{*1}}\left(w_X(z)\right)\right)}{w_X(z)} \quad (30)$$

To derive the total PGF of the system size distribution, the following generating functions are considered.

Let

$P_{Comp}\,(z)$ =The PGF of the system size when the server is busy or in breakdown state

$$=P_1^{0^*}(z,0)+BR_1^{0^{**1}}(z,0,0)$$

$$+\sum_{j=1}^{m-1}\left(P_1^{j^*}(z,0)+BR_1^{j^{**1}}(z,0,0)\right)$$

$$+\sum_{j=0}^{m-1}\sum_{i=1}^{C}\left(P_2^{(i,j)^*}(z,0)+BR_2^{(i,j)^{**1}}(z,0,0)\right)$$

Using equations (26) to (30)

$$P_{Comp}\,(z) = \frac{PIz\phi(z)}{z-\phi(z)} \quad (31)$$

Thus the total PGF of the system size distribution $P(z)$ is given by,

$$P(z) = PI + P_{Comp}\,(z)$$

$$=\frac{PI(z-1)\phi(z)}{z-\phi(z)} \quad (32)$$

PI can be calculated by using the normalizing condition P (1) =1 and found to be

$$PI = 1-\rho \quad (33)$$

where

$$\rho =$$

$$\lambda E(X)\left[\sum_{i=1}^{C}r_iE\left(H_2^{(i,s)}\right)+E\left(H_1^s\right)\right]\sum_{j=0}^{m-1}\prod_{s=0}^{j}f_s \quad (34)$$

Then (32) becomes,

$$P(z)=\frac{(1-\rho)(z-1)\phi(z)}{z-\phi(z)} \quad (35)$$

The measures E (H)'s are obtained from the LST of random variables,

$H_1^{s^*}(z) = S^*\left(ha_1^s\left(w_X(z)\right)\right)$ and

$H_2^{(i,s)^*}(z) = S_i^*\left(ha_2^{(i,s)}\left(w_X(z)\right)\right)$

$(1 \le i \le C$ and $0 \le s \le j, 0 \le j \le m-1)$

$$E(H_1^s) = E(S)\left(1+a_1^sE(R_1^s)\right) \quad (36.1)$$

$$E\left(H_2^{(i,s)}\right) = E(S_i)\left(1+a_2^{(i,s)}E\left(R_2^{(i,s)}\right)\right) \quad (36.2)$$

$$E(H_1^s)^2$$

$$=E(S)a_1^sE\left((R_1^s)^2\right)+E(S^2)\left(1+a_1^sE(R_1^s)\right)^2 \quad (36.3)$$

$$E\left(H_2^{(i,s)}\right)^2$$

$$= E(S_i)a_2^{(i,s)}E\left(\left(R_2^{(i,s)}\right)^2\right)$$

$$+E\left(S_i^{\,2}\right)\left(1+a_2^{(i,s)}E\left(R_2^{(i,s)}\right)\right)^2 \qquad (36.4)$$

E. Queue Size Distribution at Departure Epoch

If π_n^+ denotes the probability that there are n customers in the system at departure epoch, then

$$\pi_n^+ = D_1\left[\left(1-f_{j+1}\right)\sum_{i=1}^{C}\sum_{j=0}^{m-1}\left(P_{2,n+1}^{(i,j)}(0)\right)\right]$$

with normalizing constant D_1.

The PGF $\pi^+(z)$ of the queue size distribution $\{\pi_n^+ : n \geq 0\}$ at departure epoch is given by

$$\pi^+(z) = \sum_{n=0}^{\infty}\pi_n^+ z^n$$

$$= D_1\sum_{n=0}^{\infty}\left[\left(1-f_{j+1}\right)\sum_{i=1}^{C}\sum_{j=0}^{m-1}\left(P_{2,n+1}^{(i,j)}(0)\right)\right]z^n$$

$$= \frac{D_1}{z}\left[\left(1-f_{j+1}\right)\sum_{i=1}^{C}\left(P_2^{(i,0)}(z,0)\right)\right]$$

$$= \frac{D_1}{z-1}P(z)\lambda(X(z)-1)$$

Evaluating D_1 using normalizing condition,

$$\pi^+(z) = \frac{(X(z)-1)}{E(X)(z-1)}P(z)$$

F. Performance Measures:

(i) The probability that the server is busy is

$$P_{busy} = \sum_{j=0}^{m-1}P_1^j + \sum_{i=1}^{C}\sum_{j=0}^{m-1}P_2^{(i,j)}$$

where

$$\sum_{j=0}^{m-1}P_1^j = \lim_{z\to 1}P_1^{j*}(z,0) = \lambda E(X)E(S)\sum_{j=0}^{m-1}\prod_{l=0}^{j}f_l$$

$$\sum_{i=1}^{C}\sum_{j=0}^{m-1}P_2^{(i,j)} = \lim_{z\to 1}P_2^{(i,j)^*}(z,0)$$

$$= \lambda E(X)\sum_{i=1}^{C}r_i E(S_i)\sum_{j=0}^{m-1}\prod_{l=0}^{j}f_l$$

(ii) The probability that the server is in breakdown state is

$$P_{br} = \sum_{j=0}^{m-1}P_{br_1}^j + \sum_{i=1}^{C}\sum_{j=0}^{m-1}P_{br_2}^{(i,j)}$$

where

$$\sum_{j=0}^{m-1}P_{br_1}^j = \lim_{z\to 1}BR_1^{j**1}(z,0,0) = a_1^j P_1^j E\left(R_1^j\right)$$

$$\sum_{i=1}^{C}\sum_{j=0}^{m-1}P_{br_2}^{(i,j)} = \lim_{z\to 1}BR_2^{(i,j)**1}(z,0,0)$$

$$= a_2^{(i,j)}E\left(R_2^{(i,j)}\right)P_2^{(i,j)}$$

G. Mean System Size:

(iii) The expected system size of the model is given by

$$L = \left(\frac{d}{dz}(P(z))\right)_{z=1} = \frac{\phi''(1)}{2(1-\rho)} + \rho \qquad (37)$$

where

$$\phi''(1) = \sum_{j=0}^{m-1} \prod_{s=0}^{j} f_s \left[\lambda E(X(X-1)) \left(E(H_1^s) + \sum_{i=1}^{C} r_i E\left(H_2^{(i,s)}\right) \right) + \left(\lambda E(X)\right)^2 \left(\begin{array}{c} E(H_1^s)^2 + \sum_{i=1}^{C} r_i E\left(H_2^{(i,s)}\right)^2 \\ +2E(H_1^s) \sum_{i=1}^{C} r_i E\left(H_2^{(i,s)}\right) \end{array} \right) \right]$$

where $E(H_1^s), E\left(H_2^{(i,s)}\right), E(H_1^s)^2, E\left(H_2^{(i,s)}\right)^2$ are given by the equations (36.1), (36.2), (36.3) and (36.4) respectively.

H. Numerical Analysis:

In order to explore the sensitivity of the performance indices with respect to changes in the parameters of the system, numerical analysis is performed. The results are exhibited in the following graphs. The probabilities r_1, r_2, r_3, f_1, f_2 and f_3 chosen for numerical calculations are $r_1 = 0.5, r_2 = 0.4, r_3 = 0.1$, $f_1 = 0.2, f_2 = 0.5$ and $f_3 = 0.1$. The arrival rates considered are $\lambda = 0.1, 0.11, 0.12, 0.13$.

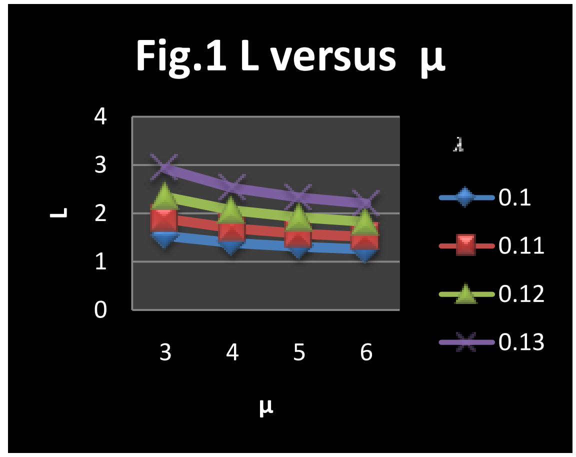
In Fig.1, the effect of first phase service rate μ with respect to the other service rates $\mu_1 = 3, \mu_2 = 5$ and $\mu_3 = 6$ is considered .

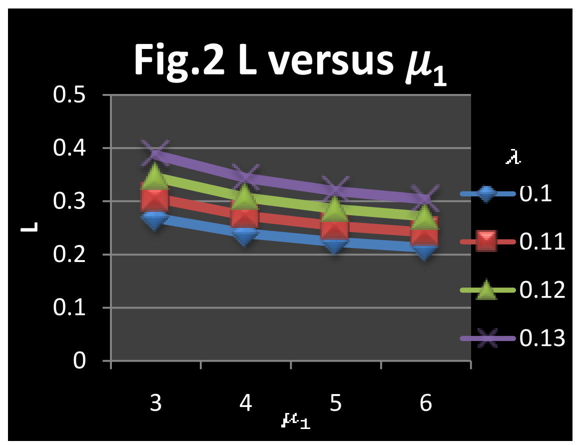
In Fig. 2, the behaviour of the system length due to changes in the service rate μ_1 (the first optional service in the second phase) is depicted for the given values $\mu = 2, \mu_2 = 5$ and $\mu_3 = 6$.

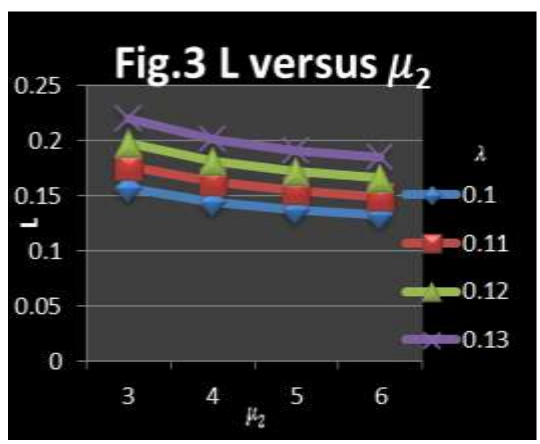
In Fig. 3, the behaviour of the expected system length due to changes in the service rate μ_2 of the second optional service in the second phase is noted for the values of $\mu = 2, \mu_1 = 3$ and $\mu_3 = 6$.

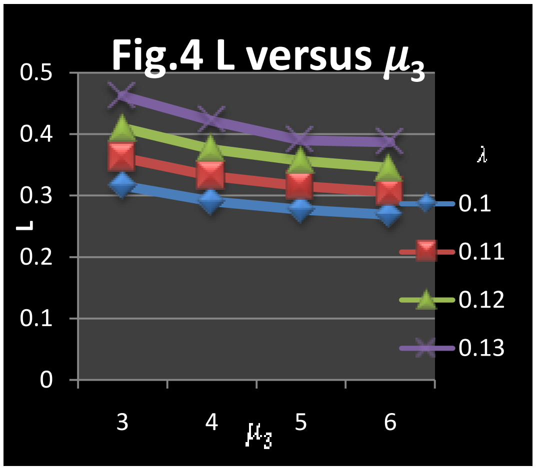
In Fig. 4, the behaviour of the expected system length due to changes in the service rate μ_3 of the third optional service in the second phase is considered for $\mu = 2, \mu_1 = 3$ and $\mu_2 = 5$.

In all the cases the queue length decreases with increasing values of μ_i 's.









Conclusion: In this paper, a batch arrival queue with two phases of service and a finite number of immediate Bernoulli feedbacks is considered for an unreliable server. The results of the present model generalise the results of different models including the model of Kalidass and Kasturi (2013) and the corresponding infinite feedback queueing models. This model is best suited for studying the banking transactions through an ATM.

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