A Note on Fuzzy (Λ, δ)-Closed Sets

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Abstract— *The motive of this article is to introduce the notions of Fuzzy (Λ, δ)-closed set and Fuzzy (Λ, δ)-continuity in fuzzy topological spaces and to investigate some basic yet essential properties.*

Keywords— *fuzzy δ-closed sets, $F\Lambda_\delta$-set, fuzzy (Λ, δ)-closed sets and fuzzy (Λ, δ)-continuity*

I. INTRODUCTION

The classical paper of L. A. Zadeh[7] in the year 1965, comprises of the concepts of fuzzy sets and fuzzy set operations. Thereafter the paper of C. L. Chang[2] in 1968 paved the way for tremendous growth of the numerous fuzzy topological concepts.

K. K. Azad[1] introduced the concept of fuzzy regular open sets and fuzzy regular closed sets in fuzzy topological spaces. Z. Petricevic[4] introduced the concept of fuzzy δ-open sets and fuzzy δ-closed sets in fuzzy topological spaces. In 2004, D.N. Georgiou[3] presented the notion of (Λ, δ)-closed sets in general topology. Thereafter this notion grasped higher significance due its nature of being partially δ-open and partially δ-closed. This work is an extension of (Λ, δ)-closed sets to fuzzy topology.

II. PREREQUISITES

Definition 2.1 : A fuzzy subset A of a fuzzy topological space (X, $\mathcal{F}$) is called

(i) **fuzzy regular open**[1] if int(cl(A))=A.

(ii) **fuzzy δ-open**[4] if $A = \bigvee_{i\in I} A_i$, where A_i is a fuzzy regular open set for each i in (X, $\mathcal{F}$).

A fuzzy subset A of a fuzzy topological space (X, $\mathcal{F}$) is called fuzzy regular closed[1](resp. fuzzy δ-closed[4]) if **1**-A is fuzzy regular open(resp. fuzzy δ-open).

Let the family of all fuzzy regular open, fuzzy regular closed, fuzzy δ-open and fuzzy δ-closed sets be represented by FRO(X, $\mathcal{F}$), FRC(X, $\mathcal{F}$), FδO(X, $\mathcal{F}$) and FδC(X, $\mathcal{F}$) respectively.

Definition 2.2 :[4] The δ-closure of a fuzzy set A is the intersection of all fuzzy regular closed sets containing A(shortly, $Cl_\delta(A)$). A fuzzy point $y_t \in Cl_\delta(A)$ iff every fuzzy regular open set which is q-coincident with x_t is also q-coincident with A.

Definition 2.3:[4] A fuzzy mapping $f : X \to Y$ is called **fuzzy super-continuous** if $f^{-1}(V)$ is a fuzzy δ-open set on X for any fuzzy open set V on Y.

III. FΛδ-SETS AND FVδ-SETS

Definition 3.1 : A fuzzy subset $F\Lambda_\delta(A)$ of a fuzzy topological space (X, $\mathcal{F}$) is defined as

$$F\Lambda_\delta(A) = \wedge \{D \in F\delta O(X, \mathcal{F}) \mid A \leq D\}.$$

Definition 3.2 : A fuzzy subset A of a fuzzy topological space (X, $\mathcal{F}$) is called a **$F\Lambda_\delta$-set** if $F\Lambda_\delta(A)=A$.

Theorem 3.3 : For fuzzy subsets A, B and A_i ($i \in I = [0,1]$) of a fuzzy topological space $(X, \mathcal{F})$, the following are true.

(i) $A \leq F\Lambda_\delta(A)$.
(ii) $F\Lambda_\delta(F\Lambda_\delta(A)) = F\Lambda_\delta(A)$.
(iii) If $A \leq B$ then $F\Lambda_\delta(A) \leq F\Lambda_\delta(B)$.
(iv) $F\Lambda_\delta(\bigwedge_{i\in I} \{A_i\}) \leq \bigwedge_{i\in I} \{F\Lambda_\delta(A_i)\}$.
(v) $F\Lambda_\delta(\bigvee_{i\in I} \{A_i\}) = \bigvee_{i\in I} \{F\Lambda_\delta(A_i)\}$.

Proof : (i), (ii) and (iii) follow from Definition 3.1.

(iv) Suppose that $x \notin \bigwedge_{i\in I} \{F\Lambda_\delta(A_i)\}$ then there exists $i_0 \in I$ such that $x \notin F\Lambda_\delta(A_{i0})$. This implies that there exists a fuzzy δ-open set D such that $x \notin D$ and $A_{i0} \leq D$. Since $\bigwedge_{i\in I} A_i \leq A_{i0} \leq D$ and $x \notin D$, we have $x \notin F\Lambda_\delta(\bigwedge_{i\in I} \{A_i\})$.

(vi) From (i) and (iii), $A_i \leq F\Lambda_\delta(A_i) \leq F\Lambda_\delta(\bigvee_{i\in I} A_i)$, for each $i \in I$. This implies $\bigvee_{i\in I} (F\Lambda_\delta(A_i)) \leq F\Lambda_\delta(\bigvee_{i\in I} A_i)$. Conversely, suppose that $x \notin \bigvee_{i\in I} (F\Lambda_\delta(A_i))$. Then $x \notin F\Lambda_\delta(A_i)$, for all $i \in I$. This implies that there exists $S_i \in F\delta O(X, \mathcal{F})$ such that $A_i \leq S_i$ and $x \notin S_i$, for all $i \in I$. Since $\bigvee_{i\in I} A_i \leq \bigvee_{i\in I} S_i$ and $\bigvee_{i\in I} S_i$ is a fuzzy δ-open set not containing x. Hence $x \notin F\Lambda_\delta(\bigvee_{i\in I} \{A_i\})$. Thus $F\Lambda_\delta(\bigvee_{i\in I} A_i\}) = \bigvee_{i\in I} \{F\Lambda_\delta(A_i)\}$.

Remark : The following Example shows that the converse of (iv) in Theorem 3.3 is not true in general.

Example 3.4: Let $X = \{a, b\}$ and $\mathcal{F}=\{\mathbf{0}, \mathbf{1}, (0.2_a, 0.5_b), (0.5_a, 0.8_b)\}$. Then $F\delta O(X, \mathcal{F}) = \{\mathbf{0}, \mathbf{1}, (0.2, 0.5)\}$. Let $A_1 = (0.1, 0.6)$ and $A_2 = (0.3, 0.3)$. Then $A_1 \wedge A_2 = (0.1, 0.3)$. Also, $F\Lambda_\delta(A_1) = F\Lambda_\delta(A_2) = \mathbf{1}$ and $F\Lambda_\delta(A_1 \wedge A_2) = (0.2, 0.5)$ which implies $F\Lambda_\delta(A_1 \wedge A_2) \ngeq F\Lambda_\delta(A_1) \wedge F\Lambda_\delta(A_2)$.

Corollary 3.5:

(i) $F\Lambda_\delta(A)$ is a $F\Lambda_\delta$-set.
(ii) If A is a fuzzy δ-open set, then A is a $F\Lambda_\delta$-set.

Proof :

(i) Follows from (ii) of Theorem 3.3.
(ii) Follows from Definition 3.2.

Theorem 3.6 : In a fuzzy topological space $(X, \mathcal{F})$, the following are true.

(i) Arbitrary Intersection of $F\Lambda_\delta$-sets is a $F\Lambda_\delta$-set.
(ii) Arbitrary Union of $F\Lambda_\delta$-sets is a $F\Lambda_\delta$-set.

Proof:

(i) Let A_i, where $i \in I$ be $F\Lambda_\delta$-sets.

$F\Lambda_\delta(\bigwedge_{i\in I} A_i)$
$= F\Lambda_\delta(A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge \ldots)$
$\leq F\Lambda_\delta(A_1) \wedge F\Lambda_\delta(A_2) \wedge \ldots \wedge F\Lambda_\delta(A_n) \wedge \ldots$
(By Theorem 3.3 (iv))
$= A_1 \wedge A_2 \wedge \ldots \wedge A_n \wedge \ldots$
(Since each is a $F\Lambda_\delta$-set)
$= \bigwedge_{i\in I} A_i$

Also by Theorem 3.3(i),

$\bigwedge_{i\in I} A_i \leq F\Lambda_\delta(\bigwedge_{i\in I} A_i)$.

Hence arbitrary intersection of $F\Lambda_\delta$-sets is a $F\Lambda_\delta$-set.

(ii) Follows directly from Theorem 3.3(v).

Definition 3.7 : A fuzzy subset $FV_\delta(A)$ of a fuzzy topological space $(X, \mathcal{F})$ is defined as

$$FV_\delta(A) = \bigvee \{C \in F\delta C(X, \mathcal{F}) \mid C \leq A\}.$$

Definition 3.8 : A fuzzy subset A of a fuzzy topological space $(X, \mathcal{F})$ is called a $\mathbf{FV_\delta}$**-set** if $FV_\delta(A) = A$.

Theorem 3.9 : For fuzzy subsets A, B and A_i ($i \in I = [0,1]$) of a fuzzy topological space $(X, \mathcal{F})$, the following are true.

(i) $FV_\delta(A) \leq A$.
(ii) $FV_\delta(FV_\delta(A)) = FV_\delta(A)$.
(iii) If $A \leq B$ then $FV_\delta(A) \leq FV_\delta(B)$.
(iv) $FV_\delta(\bigwedge_{i\in I} \{A_i\}) = \bigwedge_{i\in I} \{FV_\delta(A_i)\}$.

(v) $FV_\delta(\bigvee_{i\in I}\{A_i\}) \geq \bigvee_{i\in I}\{FV_\delta(A_i)\}$.

(vi) If A is a fuzzy δ-closed set then A is a FV_δ-set.

(vii) $F\Lambda_\delta(\mathbf{1}\text{-}A) = \mathbf{1}\text{-}FV_\delta(A)$ and
$FV_\delta(\mathbf{1}\text{-}A) = \mathbf{1}\text{-}F\Lambda_\delta(A)$.

Proof : (i) to (vi) Similar to Theorem 3.3 and Corollary 3.5.

(vii) $\mathbf{1}\text{-}FV_\delta(A)$

$= \mathbf{1} - \bigvee\{C \mid C \in F\delta C(X, \mathcal{F}) \text{ and } C \leq A\}$

$= \bigwedge\{1\text{-}C \mid 1\text{-}C \in F\delta O(X, \mathcal{F}) \text{ and } 1\text{-}C \geq 1\text{-}A\}$

$= \bigwedge\{D \mid D \in F\delta O(X, \mathcal{F}) \text{ and } 1\text{-}A \leq D\}$

$= F\Lambda_\delta(\mathbf{1}\text{-}A)$.

Similarly, we can prove the other equality.

Corollary 3.10 : $FV_\delta(A)$ is a FV_δ-set.

Proof : Follows from (ii) of Theorem 3.9.

Definition 3.11 : A map $f : (X, \mathcal{F}) \to (Y, \zeta)$ is called a **fuzzy Λ_δ-continuous**(briefly $F\Lambda_\delta$-continuous) function if the inverse image of every fuzzy closed set in (Y, ζ) is a fuzzy Λ_δ-set in $(X, \mathcal{F})$.

Theorem 3.12 : For a map $f : (X, \mathcal{F}) \to (Y, \zeta)$, the following are equivalent.

(i) f is $F\Lambda_\delta$-continuous;

(ii) Inverse image of every fuzzy open set in (Y, ζ) is fuzzy V_δ-set in $(X, \mathcal{F})$.

Proof : Follows from Theorem 3.9(vii).

IV. FUZZY (Λ, δ)-CLOSED SETS

Definition 4.1 : A fuzzy subset A of a fuzzy topological space $(X, \mathcal{F})$ is called a **fuzzy (Λ, δ)-closed** (briefly $F(\Lambda, \delta)$-closed) set if $A = K \wedge L$, where K is a $F\Lambda_\delta$-set and L is a fuzzy δ-closed set.

The family of all fuzzy (Λ, δ)-closed sets in $(X, \mathcal{F})$ is denoted by $F(\Lambda, \delta)C(X, \mathcal{F})$.

Theorem 4.2 : The following are equivalent for a fuzzy subset A of a fuzzy topological space $(X, \mathcal{F})$.

(i) A is $F(\Lambda, \delta)$-closed;

(ii) $A = K \wedge Cl_\delta(A)$, where K is a $F\Lambda_\delta$-set;

(iii) $A = F\Lambda_\delta(A) \wedge Cl_\delta(A)$;

(iv) $A = F\Lambda_\delta(A) \wedge L$, where L is a fuzzy δ-closed set.

Proof:

(i) $\Rightarrow$ (ii) Let $A = K \wedge L$, where K is a $F\Lambda_\delta$-set and L is a fuzzy δ-closed set. Now, $A \leq L \Rightarrow Cl_\delta(A) \leq L$. Aso, $A \leq K \wedge Cl_\delta(A) \leq K \wedge L = A$. Therefore $A = K \wedge Cl_\delta(A)$.

(ii) $\Rightarrow$ (iii) Let $A = K \wedge Cl_\delta(A)$, where K is a $F\Lambda_\delta$-set. Now, $A \leq K \Rightarrow F\Lambda_\delta(A) \leq F\Lambda_\delta(K) = K \Rightarrow F\Lambda_\delta(A) \leq K$. Therefore $A \leq F\Lambda_\delta(A) \wedge Cl_\delta(A) \leq K \wedge Cl_\delta(A) = A$. Hence $A = F\Lambda_\delta(A) \wedge Cl_\delta(A)$.

(iii) $\Rightarrow$ (iv) Let $A = F\Lambda_\delta(A) \wedge Cl_\delta(A)$ and put $Cl_\delta(A) = L$. Hence $A = F\Lambda_\delta(A) \wedge L$, where L is a fuzzy δ-closed set.

(iv) $\Rightarrow$ (i) Follows from Definition 4.1.

Theorem 4.3 : Every fuzzy δ-closed(resp. $F\Lambda_\delta$-) set is a $F(\Lambda, \delta)$-closed set but not conversely.

Proof : Follows from Definition 4.1 and the fact that $\mathbf{1}$ is $F(\Lambda, \delta)$-closed(resp. fuzzy δ-closed).

Example 4.4 : Let $X = \{a, b\}$ and $\mathcal{F} = \{\mathbf{0}, \mathbf{1}, (0.2_a, 0.5_b), (0.5_a, 0.8_b)\}$. Then$(0.2_a, 0.5_b)$ is $F(\Lambda, \delta)$-closed but not fuzzy δ-closed and $(0.8_a, 0.5_b)$is $F(\Lambda, \delta)$-closed but not a $F\Lambda_\delta$-set.

Theorem 4.5 : Every fuzzy δ-dense[6] set which is also $F(\Lambda, \delta)$-closed is a $F\Lambda_\delta$-set.

Proof : Let $(X, \mathcal{F})$ be a fuzzy topological space and A be a fuzzy δ-dense as well as $F(\Lambda, \delta)$-closed set in (X, τ). Then by Theorem 4.2, $A = K \wedge Cl_\delta(A)$, where K is a $F\Lambda_\delta$-set. Since A is fuzzy δ-dense, $Cl_\delta(A)=\mathbf{1}$[6] and hence $A = K$, where K is a $F\Lambda_\delta$-set.

Theorem 4.6 : Let $(X, \mathcal{F})$ be a fuzzy topological spaces. If A is fuzzy open then cl(A) is $F(\Lambda, \delta)$-closed.

Proof : If A is fuzzy open then cl(A) is fuzzy regular closed[5] and therefore fuzzy δ-closed. Further, the proof follows from Theorem 4.3.

Theorem 4.7 : Let (X, $\mathcal{F}$) be a fuzzy topological space. Then

(i) Arbitrary intersection of F(Λ, δ)-closed sets is F(Λ, δ)-closed in (X, $\mathcal{F}$).
(ii) Arbitrary union of F(Λ, δ)-open sets is F(Λ, δ)-open in (X, $\mathcal{F}$).

Proof:

(i) Let A_i be a F(Λ, δ)-closed set for each i ∈ I. Then $A_i = K_i \wedge L_i$, where K_i is a $F\Lambda_\delta$-set and L_i is a fuzzy δ-closed set for each i ∈ I. Now $\bigwedge_{i\in I} A_i = \bigwedge_{i\in I} (K_i \wedge L_i) = (\bigwedge_{i\in I} K_i) \wedge (\bigwedge_{i\in I} L_i)$. Since any intersection of $F\Lambda_\delta$-sets is a $F\Lambda_\delta$-set and fuzzy δ-closed sets is fuzzy δ-closed, $\bigwedge_{i\in I} A_i$ is a F(Λ, δ)-closed set.
(ii) Let A_i be a F(Λ, δ)-open set for each i ∈ I. Then $X \setminus A_i$ is a F(Λ, δ)-closed set for each i ∈ I. $X \setminus \bigvee_{i\in I} A_i = \bigwedge_{i\in I} (X \setminus A_i)$. Therefore by (i), $\bigvee_{i\in I} A_i$ is F(Λ, δ)-open.

Definition 4.8 : A fuzzy subset A of a fuzzy topological space (X, $\mathcal{F}$) is called a **fuzzy (Λ, δ)-open** (briefly F(Λ, δ)-open) set if $A = K \vee L$, where K is a FV_δ-set and L is a fuzzy δ-open set.

Equivalently, the complement of a fuzzy (Λ, δ)-closed set is called fuzzy (Λ, δ)-open.

The family of all fuzzy (Λ, δ)-open sets in (X, $\mathcal{F}$) is denoted by F(Λ, δ)O(X, $\mathcal{F}$).

Theorem 4.9 : The following are equivalent for a fuzzy subset A of a fuzzy topological space (X, $\mathcal{F}$).

(i) A is F(Λ, δ)-open;
(ii) $A = K \vee int_\delta(A)$, where K is a FV_δ-set;
(iii) $A = FV_\delta(A) \vee int_\delta(A)$;
(iv) $A = FV_\delta(A) \vee L$, where L is a fuzzy δ-open set.

Proof : Similar to Theorem 4.2.

Definition 4.10 : Fuzzy (Λ, δ)-closure (briefly F(Λ, δ)cl(A)) of a fuzzy subset A is defined as

$$F(\Lambda, \delta)cl(A) = \wedge \{D \in F(\Lambda, \delta)C(X, \mathcal{F}) \mid A \le D\}.$$

Theorem 4.11 : For fuzzy subsets A and B of a fuzzy topological space (X, $\mathcal{F}$), the following conditions are true.

(i) A ≤ F(Λ, δ)cl(A).
(ii) If A ≤ B, then F(Λ, δ)cl(A) ≤ F(Λ, δ)cl(B).
(iii) F(Λ, δ)cl(**0**) = **0** and F(Λ, δ)cl(**1**) = **1**.
(iv) F(Λ, δ)cl(A) is a fuzzy (Λ, δ)-closed set.
(v) A is fuzzy (Λ, δ)-closed iff F(Λ, δ)cl(A) = A.

Proof : Straight forward.

Definition 4.12 : A function f: X →Y is said to be **fuzzy (Λ, δ)-continuous**(briefly F(Λ, δ)-continuous) function if $f^{-1}(B)$ is a F(Λ, δ)-closed in X for each fuzzy closed set[4] B in Y.

Theorem 4.13 : If a fuzzy function f: X →Y is said to be fuzzy (Λ, δ)-continuous then for each fuzzy set A in X, f(F(Λ, δ)cl(A)) ≤ cl(f(A)).

Proof : cl(f(A)) is fuzzy closed in Y. By hypothesis, $f^{-1}(cl(f(A)))$ is F(Λ, δ)-closed in X. Now, $f(A) \le cl(f(A)) \Rightarrow A \le f^{-1}(f(A)) \le f^{-1}(cl(f(A))) \Rightarrow F(\Lambda, \delta)cl(A) \le F(\Lambda, \delta)cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A))) \Rightarrow f(F(\Lambda, \delta)cl(A)) \le cl(f(A))$.

Proposition 4.14 : Every fuzzy super-continuous(resp. $F\Lambda_\delta$-continuous) function is F(Λ, δ)-continuous but not conversely.

Proof : Follows from Theorem 4.3.

Example 27 : Let X = Y = {a, b} and $\mathcal{F}$ =ζ={{**0**, **1**, $(0.2_a, 0.5_b)$, $(0.5_a, 0.8_b)$}.

Define f : (X, $\mathcal{F}$) → (Y, ζ) as follows:

$$f(a,b) = \begin{cases} (a,b), & \text{if } a = 0.2 \text{ and } b = 0.5 \\ (b,a), & \text{otherwise.} \end{cases}$$

Then $f^{-1}\{(0.2_a, 0.5_b)\}=(0.2_a, 0.5_b)$ is F(Λ, δ)-open but not fuzzy δ-open. Hence f is F(Λ, δ)-continuous but not fuzzy super-continuous.

Example 28 : Let X = Y = {a, b} and $\mathcal{F}$ =ζ={{**0**, **1**, $(0.2_a, 0.5_b)$, $(0.5_a, 0.8_b)$}.

Define f : (X, $\mathcal{F}$) → (Y, ζ) as follows:

$$f(a,b) = \begin{cases} (b,a), & \text{if } a = 0.5 \text{ and } b = 0.8 \\ \mathbf{1}, & \text{otherwise.} \end{cases}$$

Then $f^{-1}\{(0.5_a, 0.8_b)\}=(0.8_a, 0.5_b)$ is F(Λ, δ)-open but not a $F\Lambda_\delta$-set. Hence f is F(Λ, δ)-continuous but not $F\Lambda_\delta$-continuous.

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