Functions Related To β* - Closed Sets in Topological Spaces

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Abstract

The purpose of this paper is to introduce Strongly β - continuous , Perfectly β* - continuous maps and basic properties and theorems are investigated.*

Keywords: *Strongly β*- continuous functions, Perfectly β* - continuous functions.*

I. Introduction

In 1960, Levine . N [7] introduced strong continuity in topological spaces. Abd El-Monsef et al. [1] introduced the notion of β-open sets and β-continuity in topological spaces. Semi-open sets, preopen sets, α-sets, and β-open sets play an important role in the researches of generalizations of continuity in topological spaces. By using these sets many authors introduced and studied various types of generalizations of continuity. In 1982, Mashhour et. al. [10] introduced preopen sets and pre-continuity in
topology. Levine [5] introduced the class of generalized closed (g-closed) sets in topological spaces. The generalized continuity was studied in recent years by Balachandran et.al. Devi et.al, Maki et.al, [3, 5]. Levine [7], Noiri [13] and Arya and Gupta introduced and investigated the concept of strongly
continuous , perfectly continuous and completely continuous functions respectively which are stronger than continuous functions. Later, Sundaram [15] defined and studied strongly g-continuous functions and perfectly g-continuous functions in topological spaces. In this paper we introduce and investigate a new class of functions called strongly β***-** continuous functions. Also we studied about β*** -** open and β*** -** closed maps and their relations with various maps.

II. Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) or X , Y , Z represent non-empty topological spaces
on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A) and int(A) denote the closure and the interior of A respectively. The power set of X is denoted by P(X).
Definition 2.1: A function f: (X, τ) ⟶ (Y, σ) is called a strongly continuous [3]if f^{-1}(O)) is both open and closed in (X, τ) for each subset O in (Y, σ).
Definition 2.2: A function f: (X, τ) ⟶ (Y, σ) is called a β - continuous if f^{-1}(O) is a β - open set of (X, τ) for every open set O of (Y, σ).
Definition 2.3: A function f: (X, τ) ⟶ (Y, σ) is called a β ***-** continuous if f^{-1}(O) is a β * - open set of (X, τ) for every open set O of (Y, σ).
Definition 2.4: A function f: (X, τ) ⟶ (Y, σ) is called a g -continuous if f^{-1}(O) is a g -open set of (X, τ) for every open set O of (Y, σ).
Definition 2.5: A function f: (X, τ) ⟶ (Y, σ) is called a perfectly continuous if f^{-1}(O) is both open and closed in (X, τ) for every open set O in (Y, σ).
Definition 2.6: A map f: (X, τ) ⟶ (Y, σ) is called a g-closed if f(O) is g-closed in (Y, σ) for every closed set O in (X, τ).
Definition 2.7: A Topological space X is said to be β *** -** $T_{1/2}$ space if every β * - open set of X is open in X.

Theorem 2.8:
(i) Every open set is β *- open and every closed set is β *-closed set
(ii) Every β -open set is β *- open and every β -closed set is β *- closed.
(iii) Every g-open set is β *-open and every g-closed set is β *-closed.

III. Strongly β * - Continuous Function

We introduce the following definition.
Definition 3.1: A function f: (X, τ) ⟶ (Y, σ) is called a strongly β *- continuous if the inverse image of every β * - open set in (Y, σ) is open in (X, τ).
Theorem 3.2: If a map f: X ⟶ Y from a topological spaces X into a topological spaces Y is strongly β *- continuous then it is continuous .
Proof: Let O be a open set in Y. Since every open set is β * - open, O is β * - open in Y. Since f is strongly β * - continuous , f^{-1}(O) is open in X. Therefore f is continuous.
Remark 3.3: The following example supports that the converse of the above theorem is not true in general.
Example 3.4: Let X = Y = {a, b, c}, τ = {ϕ, {a}, {a, b}, {a, c}, X}, σ = {ϕ, {a, b},Y}. β *O(X, τ) = {ϕ, {a}, {a, b}, {a, c}, X}, β *O(Y, σ) = {ϕ, {a}, {b},{a, b}, {a, c},{b, c}, Y}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = a, f(b) = b, f(c) = c. Clearly, f is not strongly β * - continuous. Since {b}, {b, c} is β *- open set in Y but f^{-1} ({b}) = b, f^{-1} ({b, c}) = {b, c} is not a open set of X. However, f is continuous.

Theorem 3.5: A map f: X ⟶ Y from a topological spaces X into a topological spaces Y is strongly β * - continuous if and only if the inverse image of every β *- closed set in Y is closed in X.
Proof: Assume that f is strongly β * -continuous. Let O be any β *- closed set in Y. Then O^c is β *open in Y. Since f is strongly β * - continuous, $f^{-1}(O^c)$ is open in X .But $f^{-1}(O^c)$= X/ f^{-1} (O) and so f^{-1}(O) is closed in X. Conversely, assume that the inverse image of every β *- closed set in Y is closed in X .Then O^c is β * - closed in Y. By assumption, $f^{-1}(O^c)$ is closed in X, but $f^{-1}(O^c)$ = X/ f^{-1} (O) and so f^{-1} (O) is open in X. Therefore, f is strongly β * - continuous.
Theorem 3.6: If a map f: X ⟶ Y is strongly continuous then it is strongly β * - continuous.
Proof: Assume that f is strongly continuous . Let O be any β * - open set in Y. Since f is strongly continuous, f^{-1} (O)is open in X. Therefore, f is strongly β * - continuous.
Remark 3.7: The converse of the above theorem need not be true.
Example 3.8: Let X = Y= {a, b, c}, τ = {ϕ,{b}, {c},{a, b},{b, c}, X}, τ^c ={ ϕ,{a}, {c},{a, b},{a, c}, X} and σ = {ϕ, {a},{a, b}, {a, c}, Y}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = b , f(b) = a , f(c) = c. clearly, f is strongly β *- continuous. But f^{-1} ({a}) = {b}, f^{-1} ({a, b}) = {a, b}, f^{-1} ({a, c}) = {b, c} is open in X, but not closed in X. Therefore f is not strongly continuous.
Theorem 3.9: If a map f: X ⟶ Y is strongly β * - continuous then it is β * - continuous.
Proof: Let O be an open set in Y. By [8] O is β *- open in Y. Since f is strongly β * - continuous ⇒ f^{-1} (O) is open in X. By [2.8] f^{-1} (O) is β *- open in X. Therefore, f is β *- continuous.
Remark 3.10: The converse of the above theorem need not be true.
Example 3.11: : Let X = Y= {a, b, c, d}, τ = {ϕ, {a},{a, b, c}, X} and σ = {ϕ, {a},{b, c, d},Y}, β *O(X, τ)= { ϕ, {a},{b},{c},{d},{a, b},{a, c},{a, d},{b, c},{b, d},{c, d},{a, b, c},{a, b, d},{a, c, d},{b, c, d}, X}, β *O(Y, σ)= { ϕ, {a},{b},{c},{d},{a, b},{a, c},{a, d},{b, c},{b, d},{c, d},{a, b, c},{a, b, d},{a, c, d},{b, c, d}, Y}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = a, f(b) = b, f(c) = c ,f(d) = d. Clearly, f is β * - continuous.
But f^{-1}({b})={b}, f^{-1}({c})={c}, f^{-1}({d})={d}, f^{-1}({a,b})={a,b}, f^{-1}({a,c})={a,c}, f^{-1}({a,d})={a,d}, f^{-1}({b, c})={b,c}, f^{-1}({b,d})={b,d}, f^{-1}({c,d})={c,d}, f^{-1}({a,b,d})={a,b,d}, f^{-1}({a,c,d}) = {a,c,d}, f^{-1}({b,c,d}) = {b,c,d}is not open and closed in X. Therefore f is not strongly β *- continuous.

Theorem 3.12: If a map f: X ⟶ Y is strongly β * - continuous and a map g: Y ⟶ Z is β *- continuous then g ∘ f: X ⟶ Z is continuous.

Proof: Let O be any open set in Z. Since g is β * - continuous, g^{-1} (O) is β *- open in Y . Since f is strongly β *- continuous f^{-1} (g^{-1} (O)) is open in X. But $(g \circ f)^{-1}$ (O) = f^{-1} (g^{-1} (O)). Therefore, g ∘f is continuous.

Theorem 3.13: If a map f: X ⟶ Y is strongly β * - continuous and a map g: Y ⟶ Z is β * - irresolute , then g ∘f: X ⟶ Z is strongly β * - continuous.

Proof: Let O be any β * - open set in Z. Since g is β * - irresolute, g^{-1}(O) is β * - open in Y. Also, f is strongly β * - continuous f^{-1} (g^{-1} (O)) is open in X. But $(g \circ f)^{-1}$ (O) = f^{-1} (g^{-1} (O)) is open in X. Hence, g ∘ f: X ⟶ Z is strongly β * - continuous.

Theorem 3.14: If a map f: X ⟶ Y is β * - continuous and a map g: Y ⟶ Z is strongly β * - continuous then g∘ f: X ⟶ Z is β * - irresolute.

Proof: Let O be any β * - open set in Z. Since g is strongly β * - continuous, f^{-1} (O) is open in Y. Also, f is β * - continuous, f^{-1} (g^{-1} (O)) is β * - open in X. But $(g \circ f)^{-1}$ (O) = f^{-1} (g^{-1} (O)). Hence g ∘ f: X ⟶ Z is β *- irresolute.

Theorem 3.15: Let X be any topological spaces and Y be a β * - $T_{1/2}$ space and f: X ⟶ Y be a map. Then the following are equivalent

1) f is strongly β * - continuous
2) f is continuous

Proof: (1) ⇒ (2) Let O be any open set in Y. By theorem [2.8] O is β *- open in Y. Then f -1(O) is open in X. Hence , f is continuous.

(2) ⇒ (1) Let O be any β * - open in (Y, σ). Since, (Y, σ) is a β * - $T_{1/2}$ space, O is open in (Y, σ). Since, f is continuous. Then f^{-1} (O) is open in (X, τ) . Hence, f is strongly β * - continuous.

Theorem 3.16: Let f: (X, τ) ⟶ (Y, σ) be a map. Both (X, τ) and (Y, σ) are β * - $T_{1/2}$ space. Then the following are equivalent.

1) f is β *- irresolute
2) f is strongly β *- continuous
3) f is continuous
4) f is β * - continuous

Proof: The proof is obvious.

Theorem 3.17: The composition of two strongly β * - continuous maps is strongly β * - continuous.

Proof: Let O be a β * - open set in (Z, η). Since, g is strongly β * - continuous, we get g^{-1} (O) is open in (Y, σ). By theorem [2.8] g^{-1} (O) is β * - open in (Y, σ). As f is also strongly β * - continuous, f^{-1} (g^{-1}(O)) = $(g \circ f)^{-1}$ (O) is open in (X, τ). Hence, (g ∘ f) is strongly β * - continuous.

Theorem 3.18: If f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z, η) be any two maps .Then their composition g ∘ f : (X, τ) ⟶ (Z, η) is strongly β * - continuous if g is strongly β * - continuous and f is continuous.

Proof: Let O be a β * - open in (Z, η). Since, g is strongly β * - continuous, f^{-1} (O) is open in (Y, σ). Since f is continuous , f^{-1} (g^{-1} (O)) = $(g \circ f)^{-1}$ (O) is open in (X, τ). Hence, (g ∘ f) is strongly β * - continuous.

IV. Perfectly β *- Continuous Function

Definition 4.1: A map f: (X, τ) ⟶ (Y, σ) is said to be perfectly β * - continuous if the inverse image of every β * - open set in (Y, σ) is both open and closed in (X, τ) .

Theorem 4.2: If a map f: (X, τ) ⟶ (Y, σ) from a topological space (X, τ) into a topological space (Y, σ) is perfectly β * - continuous then it is strongly β * - continuous.

Proof: Assume that f is perfectly β * - continuous. Let O be any β * - open set in (Y, σ) . Since, f is perfectly β * - continuous, f^{-1} (O) is open in (X, τ) . Therefore, f is strongly β * - continuous.

Remark 4.3: The converse of the above theorem need not be true.

Example 4.4: Let X = Y= {a, b, c}, τ = {ϕ,{c},{a, c},{b, c}, X}, τ^c = { ϕ,{a},{b},{a, b}, X}, β *O((X, τ) = {ϕ,{c},{a, c},{b, c}, X}, β *C((X, τ) = { ϕ,{a},{b},{a, b}, X}, and σ = {ϕ, {a}, {a, b}, {a, c}, Y}, β *O((Y, σ) = {ϕ,{a},{a, b},{a, c}, Y}. Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = c, f(b) = b, f(c) = a. Clearly, f is strongly β *- continuous. But f^{-1} {(a)} = {c}, f^{-1} {(a, b)} = {b, c}, f^{-1} {(a, c)} = {a, c} is open in X, but not closed in X. Therefore f is not perfectly β * - continuous.

Theorem 4.5: If a map f: (X, τ) ⟶ (Y, σ) from a topological space (X, τ) into a topological space (Y, σ) is perfectly β * - continuous then it is perfectly continuous .

Proof: Let O be an open set in Y. By theorem [2.8] O is an β * - open set in (Y, σ). Since f is perfectly β *- continuous, f^{-1} (O) is both open and closed in (X, τ) . Therefore, f is perfectly continuous.

Remark 4.6 : The converse of the above theorem need not be true.

Example 4.7: Let X ={a, b, c}, Y= {a, b, c, d}, τ = {ϕ,{a},{b, c, d}, X}, τ^c = {ϕ,{a},{b, c, d}, X} and σ = {ϕ, {a},Y}. β * O(Y, σ) = {ϕ,{a}, {b},{c},{a, b},{a, c},{b, c}, Y} Let f: (X, τ) ⟶(Y, σ) be defined by f(a) = a , f(b) = b , f(c) = c. clearly, f is perfectly continuous. But the inverse image of β * -open set in (Y, σ) [f^{-1} ({b}) = {b}, f^{-1} ({c}) = {c}, f^{-1} ({a, b}) = {a, b},f^{-1} ({a, c}) = {a, c}, f^{-1} ({b, c}) = {b, c}] is not open and closed in X. Therefore f is not perfectly β *- continuous.

Theorem 4.8: A map f: (X, τ) ⟶ (Y, σ) from a topological space (X, τ) into a topological space (Y, σ) is perfectly β *- continuous if and only if f^{-1} (O) is both open and closed in (X, τ) for every β * - closed set O in (Y, σ).

Proof: Let O be any β * - closed set in (Y, σ).Then O^c is β * - open in (Y, σ). Since, f is perfectly β * - continuous , f^{-1} (O^c) is both open and closed in (X, τ) . But f^{-1} (O^c) = X/ f^{-1} (O) and so f^{-1} (O) is both open and closed in (X, τ) . Conversely, assume that the inverse image of every β * - closed set in (Y, σ) is both open and closed in (X,τ). Let O be any β * - open in (Y, σ).Then O^c is β * - closed in (Y, σ).By assumption f^{-1}(O^c) is both open and closed in (X, τ) . But f^{-1} (O^c) = X/ f^{-1} (O) and so f^{-1} (O) is both open and closed in (X, τ).Therefore, f is perfectly β * - continuous.

Theorem 4.9: Let (X, τ) be a discrete topological space and (Y, σ) be any topological space. Let f: (X, τ) ⟶(Y, σ) be a map, then the following statements are true.

1) f is strongly β * - continuous

2) f is perfectly β * - continuous

Proof: (1) ⇒ (2) Let O be any β * - open set in (Y, σ) . By hypothesis, f^{-1} (O) is open in (X, τ). Since (X, τ) is a discrete space, f^{-1} (O) is closed in (X, τ). f^{-1} (O) is both open and closed in (X, τ). Hence, f is perfectly β * - continuous.

(2) ⇒ (1) Let O be any β * - open set in (Y, σ). Then, f^{-1} (O) is both open and closed in (X, τ). Hence, f is strongly β * - continuous.

Theorem 4.10: If f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z, η) are perfectly β * - continuous , then their composition g ∘ f : (X, τ) ⟶ (Z, η) is also perfectly β * - continuous .

Proof: Let O be a β * - open set in (Z, η) . Since, g is perfectly β * - continuous. We get that g^{-1} (O) is open and closed in (Y, σ). By theorem [2.8], g^{-1} (O) is β * - open in (Y, σ). Since f is perfectly β *- continuous, f^{-1} (g^{-1} (O)) = (g ∘ f) $^{-1}$ (O) is both open and closed in (X, τ) . Hence, g ∘ f is perfectly β * - continuous.

Theorem 4.11: If f: (X, τ) ⟶ (Y, σ) and g: (Y, σ) ⟶ (Z, η) be any two maps . Then their composition is strongly β * - continuous if g is perfectly β * - continuous and f is continuous.

Proof: Let O be any β * - open set in (Z, η). Then, g^{-1} (O) is open and closed in (Y, σ). Since, f is continuous. f^{-1} (g^{-1} (O)) = (g ∘ f)$^{-1}$ (O) is open in (X, τ) . Hence, g ∘ f is strongly β * - continuous.

Theorem 4.12: If a map f: (X, τ) ⟶ (Y, σ) is perfectly β * - continuous and a map g: (Y, σ) ⟶ (Z, η) is strongly β * - continuous then the composition g ∘ f : (X, τ) ⟶ (Z, η) is perfectly β *- continuous.

Proof: Let O be any β * - open set in (Z, η). Then, g^{-1} (O) is open in (Y, σ).By theorem [2.8] g^{-1} (O) is β * - open in (Y, σ).By hypothesis, f^{-1} (g^{-1} (O)) = (g ∘ f)$^{-1}$ (O) is both open and closed in (X, τ) . Therefore, g ∘ f is perfectly β * - continuous.

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