On Fuzzy b# Closed sets

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Abstract: In this paper, we have introduced a new class of fuzzy sets called fuzzy $b^{\#}$ closed sets, and investigated some of their properties. Some characterizations of the fuzzy $b^{\#}$closed sets are also studied.

Keywords: *Fuzzy sets, fuzzy topology, fuzzy b closed sets, fuzzy $b^{\#}$ closed sets.*

1.Introduction

The concept of fuzzy sets and fuzzy set operations were introduced by L.A.Zadeh[13]. In 1968, C.L.Chang[5] introduced the concept of fuzzy topological space which is a generalization of topological spaces. In this paper we have introduced a new type of fuzzy closed set called fuzzy $b^{\#}$closed sets and investigated some of their properties.

2.Preliminaries

Definition 2.1: [13] Let X be a non-empty set . A fuzzy set A in X is characterized by its membership function $\mu_A : X \to [0,1]$ and $\mu_A(x)$ is interpreted as the degree of member of element x in a fuzzy set A, for each $x \in X$. It is clear that A is determined by the set of tuples of $A = \{(x,\mu_A(x)) : x \in X\}$.

Definition 2.2: [13] Let $A = \{(x,\mu_A(x)) : x \in X\}$ and $B = \{(x,\mu_B(x)) : x \in X\}$ be two fuzzy sets. Then, their union $A \vee B$, intersection $A \wedge B$ and the complement A^c are also fuzzy sets with membership functions defined as follows :

(a) $\mu_A{}^c(x) = 1-\mu_A(x), \forall x \in X$,
(b) $\mu_{A\vee B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X$,
(c) $\mu_{A\wedge B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$.

Further,

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$,
(b) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$,$\forall x \in X$.

Definition 2.3: [8] A family τ of fuzzy sets is called fuzzy topology (FT in short) for X if it satisfies the three axioms:

(a) $\bar{0},\bar{1} \in \tau$
(b) $\forall A, B \in \tau \Rightarrow A \wedge B \in \tau$
(c) $\forall (A_j)_{j\in J} \in \tau \Rightarrow \vee_{j\in J} A_j \in \tau$

The pair (X, τ) is called a fuzzy topological space (FTS). The elements of τ are called fuzzy open sets (FOS) in X and their respective complements are called fuzzy closed sets (FCS) of (X, τ).

Definition 2.4: [2] A fuzzy set A in a fuzzy topological space (X, τ) is said to be a

(a) Fuzzy b closed set (FbCS) if $int(cl(A)) \wedge cl(int(A)) \leq A$
(b) Fuzzy b open set (FbOS) if $A \leq int(cl(A)) \vee cl(int(A))$

Definition 2.5: [3] Let A be a fuzzy set in a fuzzy topological space X. Then we define b-interior and b-closure as

(a) b-cl(A) = $\wedge$ { B: B $\geq$A, B is fuzzy b closed in X }
(b) b-int(A)= $\vee$ {B: B $\leq$ A, B is fuzzy b open in X }

Definition 2.8: [12] A Fuzzy set A in a FTS (X, τ) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set B in (X, τ) such that $B < cl(A)$ that is $int(cl(A)) = \bar{0}$.

Definition 2.10: [9] A fuzzy set A is quasi-coincident with a fuzzy set B, denoted by A_qB, if there exists $x \in X$ such that $A(x)+B(x) > 1$.

Definition 2.11: [9] If A and B are not quasi-coincident, then we write $A_{\bar{q}}B$. $A \leq B \Leftrightarrow A_{\bar{q}}(1-B)$.

Definition 2.12: [10] A fuzzy point $\tilde{p}$ in a set X is also a fuzzy set with membership function:

$$\mu_{\tilde{p}}(x) = \begin{cases} r, & \text{for } x = y \\ 0, & \text{for } x \neq y \end{cases}$$

where $x \in X$ and $0 < r \leq 1$, y is called the support of $\tilde{p}$ and r, the value of $\tilde{p}$. We denote this fuzzy point by x_r or $\tilde{p}$. A fuzzy point x_r is said to be belonged to a fuzzy subset $\tilde{A}$ in X, denoted by $x_r \in \tilde{A}$ if and only if $r \leq \mu_{\tilde{A}}(x)$.

3.FUZZY b# CLOSED SETS

In this section we have introduced fuzzy $b^{\#}$ closed sets and studied some of their properties.

Definition 3.1: A fuzzy set A in a FTS (X, τ) is said to be a fuzzy $b^{\#}$closed set ($Fb^{\#}CS$) if $int(cl(A)) \wedge cl(int(A)) = A$.

The family of all $Fb^{\#}CS$s of a FTS (X, τ) is denoted by $Fb^{\#}C(X)$.

Example 3.2: Let $X = \{a, b\}$ and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.3_a, 0.3_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.5_b)\rangle$. Then (X, τ) is a FTS. Let $A = \langle x, (0.5_a, 0.5_b)\rangle$ be a fuzzy set in (X, τ). Now $int(cl(A)) \wedge cl(int(A)) = G_2 \wedge G_2{}^c = A$. Then A is a $Fb^{\#}CS$ in X.

Remark 3.3: Every FCS and every $Fb^{\#}CS$ are independent to each other in general.

Example 3.4: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X where $G_1 = \langle x, (0.4_a, 0.5_b)\rangle$ and $G_2 = \langle x, (0.3_a, 0.3_b)\rangle$. Then (X, τ) is a FTS. Here $A = \langle x, (0.4_a, 0.5_b)\rangle$ is a Fb#CS as int(cl(A)) ∧ cl(int(A)) = $G_1 \wedge G_1^c$ = A but not a FCS in X as cl(A) = $G_1^c \neq A$.

Example 3.5: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.5_a, 0.5_b)\rangle$, $G_2 = \langle x, (0.4_a, 0.4_b)\rangle$. Then (X, τ) is a FTS. Here $A = \langle x, (0.6_a, 0.6_b)\rangle$ is a FCS as cl(A) = G_2^c = A but not a Fb#CS in X as int(cl(A)) ∧ cl(int(A)) = $G_1 \neq$ A.

Theorem 3.6: Every Fb#CS is a FbCS in (X, τ) but not conversely in general.

Proof: Let A be a Fb#CS in X, then int(cl(A)) ∧ cl(int(A)) = A. Now as A ≤ A, int(cl(A)) ∧ cl(int(A)) ≤ A. Therefore A is a FbCS in (X, τ).

Example 3.7: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.6_a, 0.5_b)\rangle$. Then (X, τ) is a FTS. Let $A = \langle x, (0.5_a, 0.6_b)\rangle$ be a fuzzy set in (X, τ). Now int(cl(A) ∧ cl(int(A)) = G_1 ≤ A. Therefore A is a FbCS but not a Fb#CS in X as int(cl(A) ∧ cl(int(A)) ≠ A.

Remark 3.8: As per the above Theorem 3.6 and Example 3.7, Fb#CS is stronger than FbCS in X.

Theorem 3.9: Every FRCS [11] and every Fb#CS are independent to each other in general.

Example 3.10: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.4_a, 0.3_b)\rangle$. Then (X, τ) is a FTS. Here $A = \langle x, (0.5_a, 0.6_b)\rangle$ is a Fb#CS in X but not a FRCS in X as cl(int(A)) = $G_2^c \neq$ A.

Example 3.11: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.5_a, 0.3_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.6_b)\rangle$. Then (X, τ) is a FTS. Here $A = \langle x, (0.5_a, 0.4_b)\rangle$ is a FRCS in X as cl(int(A)) = G_2^c = A but not a Fb#CS in X as int(cl(A)) ∧ cl(int(A)) = $G_1 \neq$ A.

Theorem 3.12: Every FPCS [7] and every Fb#CS are independent to each other in general.

In the following diagram we have provided the relation between various types of fuzzy closedness.

Example 3.13: In Example 3.10, A = ⟨x, *(0.5*a, *0.6*b*)* is a Fb#CS in X but not a FPCS in X as cl(int(A)) = $G_2^c \nleq$ A.

Example 3.14: Let X = {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.5_a, 0.6_b)\rangle$, $G_2 = \langle x, (0.4_a, 0.5_b)\rangle$. Then (X, τ) is a FTS. Here $A = \langle x, (0.6_a, 0.5_b)\rangle$ is a FPCS in X as cl(int(A)) = G_2^c ≤ A but not a Fb#CS in X as int(cl(A)) ∧ cl(int(A)) = $G_2 \neq$ A.

Theorem 3.15: Every Fb#CS is a FSCS [1] in (X, τ) but not conversely in general.

Proof: Let A be a Fb#CS in X, then int(cl(A)) ∧ cl(int(A)) = A. Now as A ≤ A, int(cl(A)) = int(cl(int(cl(A)))) ∧ cl(int(A)) ≤ int(cl(cl(A))) ∧ cl(int(A)) ≤ int(cl(A))∧ cl(cl(int(A))) = A. Hence A is a FSCS in (X, τ).

Example 3.16: Let X = {a,b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where $G_1 = \langle x, (0.4_a, 0.3_b)\rangle$, $G_2 = \langle x, (0.5_a, 0.4_b)\rangle$. Then (X, τ) is a FTS. Here the fuzzy set $A = \langle x, (0.5_a, 0.6_b)\rangle$ is a FSCS as int(cl(A)) = G_2 ≤ A but not a Fb#CS in X as int(cl(A)) ∧ cl(int(A)) = $G_2 \neq$ A.

Theorem 3.17: Every FαCS [6] and every Fb#CS are independent to each other in general.

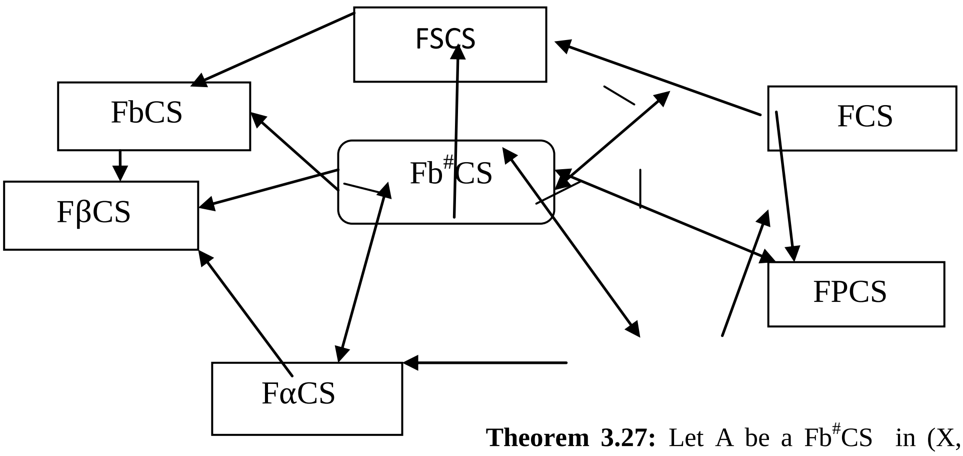
Example 3.18: In Example 3.10, A = ⟨x, *(0.5*a, *0.6*b*)* is a Fb#CS in X but not a FαCS in X as cl(int(cl(A))) = $G_2^c \nleq$ A.

Example 3.19: Let X= {a, b} and $\tau = \{\bar{0}, \bar{1}, G_1, G_2\}$ be a FT on X, where a $G_1 = \langle x, (0.5_a, 0.4_b)\rangle$, $G_2 = \langle x, (0.6_a, 0.5_b)\rangle$. Then (X, τ) is a FTS. Here the fuzzy set $A = \langle x, (0.5_a, 0.6_b)\rangle$ is a FαCS in X as cl(int(cl(A))) = G_1^c ≤ A but not a Fb#CS in X as int(cl(A)) ∧ cl(int(A)) = $G_1 \neq$ A.

Theorem 3.20: Every Fb#CS is a FβCS [4] in (X, τ) but not conversely in general.

Proof: Let A be a Fb#CS in X, then int(cl(A)) ∧ cl(int(A)) = A. Now int(cl(int(A))) = int(cl(int(A))) ∧ cl(int(A)) ≤ int(cl(A)) ∧ cl(int(A)) = A. We have int(cl(int(A))) ≤ A. Hence A is a FβCS in (X, τ).

Example 3.21: In example 3.14, $A = \langle x, (0.6_a, 0.5_b)\rangle$ is a FβCS in X but not a Fb#CS as int(cl(A)) ∧ cl(int(A)) ≠ A.



Theorem 3.22: If A is both a FROS and a FRCS then A is a Fb$^{\#}$CS in (X, τ).
Proof: Let A be both a FROS and a FRCS in (X, τ). Then $int(cl(A)) \wedge cl(int(A)) = A \wedge A = A$. This implies A is a Fb$^{\#}$CS in (X, τ).
Theorem 3.23: If A is both a FOS and a FCS then A is a Fb$^{\#}$CS in (X, τ).
Proof: Let A be both a FOS and a FCS in (X, τ). Then $int(cl(A)) \wedge cl(int(A)) = int(A) \wedge cl(A) = int(A) = A$. Therefore A is a Fb$^{\#}$CS in (X, τ).
Theorem 3.24: For a FS A in (X, τ), the following are equivalent:

(i) A is both a FOS and a Fb$^{\#}$CS.
(ii) A is a FROS.

Proof:(i)⇒(ii) Let A be a FOS and a Fb$^{\#}$CS in X. Then $A = int(cl(A)) \wedge cl(int(A)) = int(cl(A)) \wedge cl(A) = int(cl(A))$. Hence A is a FROS in X.
(ii) ⇒ (i) Let A be a FROS in X. Then $A = int(cl(A))$. Since every FROS is a FOS, A is a FOS in X. Therefore $int(cl(A)) \wedge cl(int(A)) = A \wedge cl(int(A)) = A \wedge cl(A) = A$. Hence A is a Fb$^{\#}$CS in X.
Theorem 3.25: For a Fb$^{\#}$CS A in a FTS (X, τ), the following conditions hold:

(i) If A is a FROS then scl(A) is a Fb$^{\#}$CS
(ii) If A is a FRCS then sint(A) is a Fb$^{\#}$CS

Proof:(i) Let A be a FROS in (X, τ). Then $int(cl(A)) = A$. By definition we have $scl(A) = A \vee int(cl(A)) = A$. Since A is a Fb$^{\#}$CS in X, scl(A) is a Fb$^{\#}$CS in X.
(ii) Let A be a FRCS in (X, τ). Then $cl(int(A)) = A$. By definition we have $sint(A) = A \wedge cl(int(A)) = A$. since A is a Fb$^{\#}$CS in X, sint(A) is a Fb$^{\#}$CS in X.
Theorem 3.26: If A is both a Fb$^{\#}$CS and a FCS then A is a FOS in (X, τ).
Proof: Let A be a Fb$^{\#}$CS and a FCS. Then $A = int(cl(A)) \wedge cl(int(A))$. Now $A = int(cl(A)) \wedge cl(int(A)) = int(A) \wedge cl(int(A)) = int(A)$. Hence A is a FOS in X.

Theorem 3.27: Let A be a Fb$^{\#}$CS in (X, τ) and $\mu_{\tilde{p}}(x)$ be a fuzzy point such that $\mu_{\tilde{p}}(x)_q(cl(int(A)) \wedge int(cl(A))$. Then $cl(\mu_{\tilde{p}}(x))_q A$.
Proof: Assume that A is a Fb$^{\#}$CS in (X, τ) and $\mu_{\tilde{p}}(x)_q(cl(int(A)) \wedge int(cl(A))$. Suppose that $cl(\mu_{\tilde{p}}(x))_{\bar{q}}A$, then $A \leq (cl(\mu_{\tilde{p}}(x)))^c$ where $(cl(\mu_{\tilde{p}}(x)))^c$ is a FOS in (X, τ). Then by hypothesis, $A = cl(int(A)) \wedge int(cl(A)) \leq (cl(\mu_{\tilde{p}}(x)))^c = int(\mu_{\tilde{p}}(x))^c \leq (\mu_{\tilde{p}}(x))^c$. Therefore $(cl(int(A)) \wedge int(cl(A))q(\mu p(x))$, which is a contradiction to the hypothesis. Hence $cl(\mu_{\tilde{p}}(x))_q A$.

4.FUZZY b$^{\#}$ OPEN SETS

In this section we have introduced a new type of fuzzy open set called fuzzy b$^{\#}$ open sets and studied some of its properties.
Definition 4.1 : The complement A^c of a Fb$^{\#}$CS A in a FTS (X, τ) is called a fuzzy b$^{\#}$ open set (Fb$^{\#}$OS in short) in X.
The family of all Fb$^{\#}$OSs of a FTS (X, τ) is denoted by Fb$^{\#}$O(X).
Example 4.2: In example 3.2, let $A = \langle x, (0.5a, 0.5b) \rangle$ be a FS in (X, τ). Now $cl(int(A^c)) \wedge int(cl(A^c)) = G_2 = A$, where G_2 is a FOS in X. This implies that A^c is a Fb$^{\#}$CS in X. Hence A is a Fb$^{\#}$OS in X.
Theorem 4.3: Every Fb$^{\#}$OS are FbOS, FSOS, FβOS but not conversely in general.
Proof: Straight forward.
Example 4.4: Obvious from Example 3.7, Example 3.16, Example 3.21 by taking complement of A in the respective examples.
Theorem 4.5: Every FOS, FROS, FPOS, FαOS and every Fb$^{\#}$OS in (X, τ) are independent to each other in general.
Example 4.6: Obvious from Example 3.4 and Example 3.5, Example 3.10 and Example 3.11, Example 3.13 and Example 3.14, Example 3.18 and Example 3.19, by taking complement of A in the respective examples.
Theorem 4.7: If A is b$^{\#}$ -open and nowhere dense then A is regular open in (X, τ) .

Proof: Let A is $b^{\#}$ -open and nowhere dense. Then A = int(cl(A)) ∨ cl(int(A)) = $\bar{0}$ ∨ cl(int(A)) = cl(int(A)). Therefore cl(int(A)) = A. Hence A is regular open.

Theorem 4.8: If A is both a $Fb^{\#}OS$ and a FOS then A is a FCS in (X, τ).

Proof: Let A be a $Fb^{\#}OS$ and a FOS. Then A = int(cl(A)) ∨ cl(int(A)). Now A = int(cl(A)) ∨ cl(int(A)) = int(cl(A) ∨ cl(A) = cl(A). Hence A is a FCS.

Theorem 4.9: If A is both a $Fb^{\#}OS$ and a FSCS then A is a FCS in (X, τ).

Proof: Let A be a $Fb^{\#}OS$ and a FSCS. Then A = int(cl(A)) ∨ cl(int(A)). Now A = int(cl(A)) ∨ cl(int(A)) ≤ A ∨ cl(int(A)) ≤ A ∨ cl(A) ≤ cl(A). Therefore A = cl(A). Hence A is FCS.

Theorem 4.10: Let A be a $Fb^{\#}OS$ in a FTS in X such that Int A = $\bar{0}$, then A is a FPOS in X.

Proof: Let A be a $Fb^{\#}OS$ in X. Then A ≤ int(cl(A)) ∨ cl(int(A)) ≤ int(cl(A)) ∨ $\bar{0}$ ≤ int(cl(A)). Hence A is a FPOS in X.

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