International Journal of Mathematics Trends and Technology (IJMTT)- Special Issue ICRMIT March 2018

# COM-Poisson Beta Distribution

V.Saavithri
Department of Mathematics
Nehru Memorial College
Trichy.

S.Thilagarathinam
Department of Mathematics
Nehru Memorial College
Trichy.

P.Bavithra
Department of Mathematics
Nehru Memorial College
Trichy.

**Abstract**

In this paper, we introduce COM-Poisson Beta distribution. The new COM-Poisson beta distribution density function can be expressed as a mixture of COM-Poisson and beta distribution density function. Its properties are also studied.

**Key Words:** Poisson distribution, Beta distribution, COM-Poisson Distribution, COM-Poisson Beta distribution.

## 1 Introduction

The Conway-Maxwell Poisson distribution discussed by Shmulli et.al.(2005). This distribution as an extension of the Poisson distribution is a popular model for analysing counting data. The COM-Poisson distribution with parameter $\lambda > 0$ and $\nu \geq 0$, say $COMP(\lambda, \nu)$. The case $\nu = 1$ corresponding to the Poisson distribution. The values of $\nu > 1$ correspond to under-dispersion, whereas the values of $\nu < 1$ represent over-dispersion with respect to the Poisson distribution. When $\nu$ tends to infinitely, the Com-Poisson distribution approaches to the Bernoulli distribution with parameter $(1 + \lambda)^{-1}$. For $\nu = 0$ and $\lambda < 1$, the COM-Poisson distribution reduces to the geometric distribution with parameter $1 - \lambda$.

The beta distribution of the first kind, usually written in terms of the incomplete beta function, can be used to model the distribution of measurements whose values all lie between zero and one. Gurand (1958) made the further assumption that the parameter $p$ varies from cluster to cluster. This variation can be represented by a beta distribution. The distribution of the number of cluster Gurand assumed also that the number of egg masser per plot is Poisson.

The beta binomial model for the distribution is used very widely. Muench (1963, 1938) was interested in applications to medical trials. Skellam (1948) applied the model to the association of chromosomes and to traffic clusters. Guenther (1971) showed that the average cost per lot for the Hald linear cost model with a beta prior has a beta-binomial distribution. The beta-binomial distribution has itself been used as a prior distribution by Steck and Zimmer (1968).

In this paper, we introduce COM-Poisson Beta distribution. The new COM-Poisson beta distribution density function can be expressed as a mixture of COM-Poisson and beta distribution density function. Its properties are also studied.

In section 2,3, COM-Poisson and Beta Distributions are defined. In section 4, COM-Poisson Beta Distribution is defined.Its properties are derived in section 5.

## 2 COM-Poisson Distribution

The probability density function of COM-Poisson distribution is

$$P(X = x) = \frac{\lambda^x}{(x!)^\nu} \frac{1}{Z(\lambda, \nu)}, \qquad x = 0, 1, 2, ...$$

where $Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}$ for $\lambda > 0$ and $\nu \geq 0$.

The probability generating function of COM-Poisson distribution is

$$P_X(s) = \frac{Z(\lambda s, \nu)}{Z(\lambda, \nu)}$$

.

## 3 Beta Distribution

The Beta distribution is given by

$$f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1}$$

where the parameters $\alpha$ and $\beta$ are positive real quantities and the variable $x$ satisfies $0 \leq x \leq 1$. The quantity $B(\alpha, \beta)$ is the Beta function defined in terms of the more common Gamma function as

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

For $\alpha = \beta = 1$ the Beta distribution simply becomes a uniform distribution between zero and one.

## 4 Com-Poisson Beta Distribution

The COM-Poisson beta distribution arises in a model formed by supposing that objects (which are to be countable) occur in clusters. Suppose there are $Y$ independent random variables of the form $X$, and $N$ denotes the sum of these random variables, namely

$$N = X_1 + X_2 + ...X_Y$$

Then, the COM-Poisson beta distribution model is derived by the following assumptions

(i) $X$ represents the number of objects with in a cluster where $X$ follows COM-Poisson distribution with parameters $\lambda$, $\nu$.
(ii) $Y$ represents the number of clusters, where $Y$ follows beta distribution with parameters $\alpha$, $\beta$.

This random variable $N$, formed by compounding in this fashion gives rise to the Com-poisson beta distribution and its probability generating function can be derived easily.

The probability generating function of $X$ is known to be

$$G_X(s) = \frac{Z(\lambda s, \nu)}{Z(\lambda, \nu)} \tag{1}$$

The probability generating function of $Y$ is

$$G_Y(s) = \int_0^1 s^y \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} dy$$

Since $X_i$'s are iid and independent of $Y$, the Probability generating function of the random variable $N$ can be derived as follows

$$\begin{aligned}
G_N(s) &= E(s^N) = E\left(s^{X_1+X_2+\ldots+X_y}\right) \\
&= \sum_{y=0}^{\infty} E\left(s^{X_1+X_2+\ldots+X_y|Y=y}\right) P(Y=y) \\
&= \sum_{y=0}^{\infty} [E(s)]^y\, P(Y=y) \\
&= \sum_{y=0}^{\infty} [G_X(s)]^y\, P(Y=y) = G_Y(G_X(s)) \\
&= \int_0^1 (G_X(s))^y\, \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy
\end{aligned}$$

$$G_N(s) = \int_0^1 \left(\frac{Z(\lambda s,\nu)}{Z(\lambda,\nu)}\right)^y \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy \tag{2}$$

Now, since the probability generating function of $N$ in (2) can be expressed as

$$\int_0^1 \left(\frac{Z(\lambda s,\nu)}{Z(\lambda,\nu)}\right)^y \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy = \int_0^1 \frac{\sum_{j=0}^{\infty}\frac{(\lambda y s)^j}{(j!)^\nu}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

upon collecting the coefficient of $s^m$ in the above series, we find an explicit expression for the Probability mass function of $N$ as.

$$P(N=m) = \int_0^1 \frac{(\lambda y)^m}{(Z(\lambda,\nu))^y m!} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

$$P(N=x) = \int_0^1 \frac{(\lambda y)^x}{(Z(\lambda,\nu))^y x!} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

This is the probability mass function of the Com-Pisson beta distribution and denote it by $N \sim CPB(\lambda,\nu,\alpha,\beta)$

## 5 Properties

The mean and variance can be calculated from the first and second derivatives of the probability generating function by setting $s = 1$

$$G'_N(s) = \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda ys)^{j-1}\lambda y}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

$$\text{Mean} = G'_N(1) = \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

$$G''_N(s) = \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(j-1)(\lambda ys)^{j-2}(\lambda y)^2}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

$$G''_N(1) = \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(j-1)(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy$$

$$Var(N) = G''_N(1) + G'_N(1) - [G'_N(1)]^2$$

$$Var(N) = \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(j-1)(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy + \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy - \left[ \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy \right]^2$$

From these,we find the ratio between mean and variance to be

$$\frac{Var(N)}{Mean} = \frac{\int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(j-1)(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy}{\int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy} - \int_0^1 \frac{\sum_{j=0}^{\infty} \frac{j(\lambda y)^{j}}{(j!)^{\nu}}}{(Z(\lambda,\nu))^y} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)} dy + 1$$

## References


[1] Gurland J (1958):*A generalized class of contagious distribution* Biometrics, 14, 229-249.

[2] Guether W. C (1971): *The average sample number for truncated double sample attribute plans, technometrics*, 13, 811-816.

[3] Muuench H (1938): *Discrete frequency distributions arising from mixtures of several single probabilitynvalues*, journal of American Statistical Association, 33, 390-398.

[4] Shmeli G, Minka T.P, Kadane J.B, Borle S and Boatwright P (2005): *A useful distribution for fitting discrete data:Revival of the COM-Poisson ditribution*, J.R.Stat.Soc.Ser.C(Appl. Stat), 54, 127-142.

[5] Skellam J.G (1948), *A probability distribution derived from the binomial distribution by regarding the probability of success as variable berween the sets of trials*, Journal of the Royal Statistical Society, series B, 10, 257-261.

[6] Steck G.P., and Zimmer W.J (1968) *The relationship between Neyman and Bayes confidence intervals for the hypergeometric parameter*, Technometrics, 10, 199-203.

[7] Priyadharshini J., and Saavithri,V., *COM-Poisson Polya-Aeppli Process*, Paper communicated to the journal of Statistics and probability letters.